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FILTERED COCATEGORIES

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Abstract. We recall the notions of a graded cocategory, conilpotent cocategory, morphisms of such (cofunctors), coderivations and define their analogs in $\mathbb{L}$-filtered setting. The difference with the existing approaches: we do not impose any restriction on Λ-modules of morphisms (unlike Fukaya and collaborators), we consider a wider class of filtrations than De Deken and Lowen (including directed groups $\mathbb{L}$). Results for completed filtered conilpotent cocategories include: cofunctors and coderivations with value in completed tensor cocategory are described, a partial internal hom is constructed as the tensor cocategory of certain coderivation quiver, when the second argument is a completed tensor cocategory.

Introduction. The subject of usual (non-filtered) A_∞-categories is absorbed to some extent by the subject of **dg**-categories since any non-filtered A_∞-category is equivalent to a **dg**-category. On the other hand, A_∞-categories arising in symplectic geometry (Fukaya categories) are naturally $\mathbb{R}$-filtered. Hence the necessity to study filtered A_∞-categories per se. Such a study began in works of Fukaya (*e.g.* [Fukaya, 2002]) continued in his works with Oh, Ohta and Ono (*e.g.* [Fukaya, Oh, Ohta, Ono, 2009]). The restriction imposed in these works (dictated by the geometric origin of Fukaya categories) is that the modules of morphisms are torsion free over a graded commutative filtered ring Λ, the Novikov ring. The various freeness requirements are removed in the approach of [De Deken, Lowen, 2018]. However, they work with $\mathbb{L}$-filtered modules, where the commutative monoid $\mathbb{L}$ has partial ordering such that the neutral element 0 is the smallest element of $\mathbb{L}$. For instance, such is $\mathbb{L} = \mathbb{R}_{\geqslant 0}$, but not $\mathbb{L} = \mathbb{R}$.

Here we relax the conditions on partially ordered commutative monoid $\mathbb{L}$, whose elements index the filtration, thereby including directed groups. And we keep the feature of not necessarily torsion free modules of morphisms. Thus we combine the features of works of Fukaya, Oh, Ohta and Ono on the one hand and of works of De Deken and Lowen on the other. I hope that this combination will be useful for articles on Homological Mirror Conjecture of [Kontsevich, 1995].

In the present article we deal mostly with a predecessor of A_∞-categories – $\mathbb{L}$-filtered $\mathbb{Z}$-graded cocategories $\mathfrak{A}$ over a graded commutative complete $\mathbb{L}$-filtered ring Λ. Among $\mathfrak{A}$ we distinguish conilpotent cocategories $\mathfrak{a}$, especially, tensor quivers $T\mathfrak{b}$ with cut comultiplication, and their completions $\hat{\mathfrak{a}}$ and $\widehat{T\mathfrak{b}}$, respectively. The completions are taken with respect to the uniform structure coming from the filtration. The uniform structure is one of the tools we use in the study of filtered cocategories. Our results include a

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description of morphisms (cofunctors) from a completed conilpotent cocategory $\hat{\mathfrak{a}}$ to $\widehat{T\mathfrak{b}}$. Furthermore, we describe coderivations between such cofunctors. We define a partial internal hom between completed conilpotent cocategories. Partial because the considered second arguments are only of the form $\widehat{T\mathfrak{b}}$. This internal hom is the tensor cocategory of a coderivation quiver. The latter has cofunctors as objects and coderivations as morphisms. We define also the evaluation cofunctor and prove its property which justifies the name of evaluation. Composition of cofunctors $\widehat{T\mathfrak{a}} \to \widehat{T\mathfrak{b}} \to \widehat{T\mathfrak{c}}$ extends to composition (cofunctor) of internal homs. When the source $\widehat{Ts\mathcal{A}}$ and the target $\widehat{Ts\mathcal{B}}$ are filtered A_∞-categories (equipped with a differential of degree 1 preserving the filtration), so is the coderivation quiver (up to a shift).

Contents

Plan of the article. In the first section we deal with non-filtered graded cocategories. To some extent this is a recollection of [Lyubashenko, 2003] and serves as an introduction to the filtered case. The new exposition differs from [Lyubashenko, 2003] in the use of conilpotent cocategories as a source instead of tensor categories. Also the proofs of the main results (Proposition 1.14) are new.

The second section is devoted to the main subject – $\mathbb{L}$-filtered $\mathbb{Z}$-graded cocategories, especially, to completed conilpotent cocategories. We begin with conditions on a commutative partially ordered monoid $\mathbb{L}$. We study (complete) $\mathbb{L}$-filtered $\mathbb{Z}$-graded abelian groups and later (complete) $\mathbb{L}$-filtered Λ-modules (Section 2.8), where Λ is a graded commutative complete $\mathbb{L}$-filtered ring, for instance, the universal Novikov ring. In Section 2.11 we define completed conilpotent cocategories and their morphisms (cofunctors). We describe cofunctors with values in a completed tensor cocategory in Theorem 2.23. In Section 2.30 we study coderivations, in particular, in Proposition 2.32 we describe coderivations with values in a completed tensor cocategory. In Section 2.37 we define the evaluation cofunctor and prove in Theorem 2.38 its property, which justifies the name of evaluation.

In the third section we apply these results to differential graded completed tensor cocategories also known as filtered A_∞-categories. We prove in Proposition 3.2 that the coderivation quiver for two filtered A_∞-categories is a filtered A_∞-category itself (up to a shift). Examples are given.

Conventions. We work in Tarski–Grothendieck set theory originated in [Tarski, 1939]. In this theory everything is a set (or an element of a set) and any set is an element of some Grothendieck universe. In particular, any universe is an element of some (bigger) universe.

Let $\mathcal{V}$ be a symmetric monoidal category. By a lax plain/symmetric/braided monoidal $\mathcal{V}$-category $\mathcal{C}$ we mean a $\mathcal{V}$-category equipped with $\otimes^I$, λ^f, ρ^L with the properties listed in [Bespalov, Lyubashenko, Manzyuk, 2008, Definition 2.10] (the one with natural transformations $\lambda^f : \otimes^{i\in I} M_i \to \otimes^{j\in J} \otimes^{i\in f^{-1}j} M_i$ for non-decreasing/arbitrary/arbitrary map of finite ordered sets $f : I \to J$). The same notion bears the name 'oplax' in the works of Day, Street, Leinster, Schwede, Shipley and many other authors. Any finite ordered set I is isomorphic to $\mathbf{n} = \{1 < 2 < \cdots < n\}$ for a unique $n \geqslant 0$. In order to reduce the data we assume that $\otimes^I = \left(\mathcal{C}^I \cong \mathcal{C}^n \xrightarrow{\otimes^n} \mathcal{C}\right)$; λ^f identifies with λ^g, where $f : I \to J$ is a map of finite ordered sets and $g : \mathbf{n} \to \mathbf{m}$ comes from the commutative square

$$\begin{array}{ccc} I & \xrightarrow{\cong} & \mathbf{n} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ J & \xrightarrow{\cong} & \mathbf{m} \end{array}$$

ρ^L for 1-element set L reduces to $\rho^1 : \otimes^1 \to \mathrm{Id}$. This reduction is used only for easier writing and one can get rid off it whenever needed. Similarly, in the definition of a $(\mathcal{V})$-multicategory C we assume that $\mathsf{C}((M_i)_{i\in I}; N) = \mathsf{C}(M_{\phi(1)}, \ldots, M_{\phi(n)}; N)$ for the only non-decreasing bijection $\phi : \mathbf{n} \to I$, with the corresponding requirement on compositions for C. Summing up, the notion reduces to $I \in \{\mathbf{n} \mid n \in \mathbb{Z}_{\geqslant 0}\}$ and we may simply write $\mathsf{C}(M_1, \ldots, M_n; N)$. The results of the article extend obviously to the picture indexed by arbitrary finite ordered sets, which is anyway isomorphic to the picture in which only $\mathbf{n}$ are used as indexing sets.

Composition of two morphisms of certain degrees $f : X \to Y$ and $g : Y \to Z$ is mostly denoted $fg = f \cdot g$. When the sign issues are irrelevant the composition may be denoted $gf = g \circ f$. Applying a mapping f of certain degree to an element x of certain degree we typically write $xf = (x)f$. When there are no sign issues the same may be written as $fx = f(x)$.

Acknowledgement. I am really grateful to Kaoru Ono for explaining the geometric side of Fukaya categories.

1. Conilpotent cocategories

Let $\mathcal{V}$ be the complete additive symmetric monoidal category with small coproducts and directed colimits $\mathcal{V} = \mathbf{gr} = \mathbf{gr}_\Lambda = \Lambda$-mod, where Λ is a $\mathbb{Z}$-graded commutative ring, Λ-mod means the category of $\mathbb{Z}$-graded abelian groups which are also Λ-modules (and the action $\Lambda \otimes M \to M$ has degree 0). Besides these properties sometimes we use also that $\mathcal{V}$ is closed symmetric monoidal. Examples of Λ are the universal Novikov ring $\Lambda_{0,nov}(R)$ and its localization $\Lambda_{nov}(R)$, see [Fukaya, Oh, Ohta, Ono, 2009, §1.7 (Conv. 4)]. Left Λ-modules are viewed as commutative Λ-bimodules, $\pm m\lambda \equiv \mu\tau(\lambda \otimes m) = \lambda m$. In general, commutativity is considered with respect to the symmetry $\tau(x \otimes y) = (-)^{xy} y \otimes x =$

$(-1)^{\deg x\cdot\deg y}y\otimes x$. Thus, $(\mathcal{V},\otimes,\mathbb{1},\tau,\underline{\mathcal{V}})$ means $(\mathbf{gr}_\Lambda,\otimes_\Lambda,\Lambda,\tau,\underline{\mathbf{gr}})$, where the inner hom is

$$\underline{\mathbf{gr}}(M,N)^d=\{f\in\prod_{n\in\mathbb{Z}}\mathsf{Ab}(M^n,N^{n+d})\mid\forall n\in\mathbb{Z}\,\forall p\in\mathbb{Z}\,\forall\lambda\in\Lambda^p\ \lambda f=f\lambda:M^n\to N^{n+d+p}\}.$$

1.1. Definition. *A $\mathcal{V}$-quiver $\mathfrak{a}$ is a set of objects $\operatorname{Ob}\mathfrak{a}$ and an object $\mathfrak{a}(X,Y)\in\operatorname{Ob}\mathcal{V}$ given for each pair of objects $X,Y\in\operatorname{Ob}\mathfrak{a}$. The category of $\mathcal{V}$-quivers $\mathcal{V}$-Quiv has as morphisms $f:\mathfrak{a}\to\mathfrak{b}$ collections consisting of a map $f=\operatorname{Ob}f:\operatorname{Ob}\mathfrak{a}\to\operatorname{Ob}\mathfrak{b}$ and morphisms $f:\mathfrak{a}(X,Y)\to\mathfrak{b}(fX,fY)$ for each pair of objects $X,Y\in\operatorname{Ob}\mathfrak{a}$.*

1.2. Example. Let S be a set. One forms a $\mathcal{V}$-quiver $\mathbb{1}S$ with $\operatorname{Ob}\mathbb{1}S=S$,

$$\mathbb{1}S(X,Y)=\begin{cases}\mathbb{1}, & \text{if } X=Y,\\ 0, & \text{if } X\neq Y.\end{cases}$$

A mapping $f:S\to Q$ induces a quiver morphism $\mathbb{1}f:\mathbb{1}S\to\mathbb{1}Q$ with $\operatorname{Ob}\mathbb{1}f=f$ and $\mathbb{1}f=\mathrm{id}_{\mathbb{1}}:\mathbb{1}S(X,X)\to\mathbb{1}Q(fX,fX)$ for any $X\in S$.

The category $\mathcal{V}$-Quiv is symmetric monoidal with the tensor product $\boxtimes$

$$\operatorname{Ob}\mathfrak{a}\boxtimes\mathfrak{b}=\operatorname{Ob}\mathfrak{a}\times\operatorname{Ob}\mathfrak{b},$$
$$(\mathfrak{a}\boxtimes\mathfrak{b})((A,B),(A',B'))=\mathfrak{a}(A,A')\otimes\mathfrak{b}(B,B').$$

The unit object quiver $\mathbb{1}$ has one-element set $\operatorname{Ob}\mathbb{1}=\{*\}$ and $\mathbb{1}(*,*)=\mathbb{1}$. The symmetry comes from that of $\mathcal{V}$. Since the symmetric monoidal category $\mathcal{V}$ is closed, so is $\mathcal{V}$-Quiv with inner hom object $\underline{\mathcal{V}\text{-}\mathsf{Quiv}}(\mathfrak{a},\mathfrak{b})$, which is the $\mathcal{V}$-quiver with $\operatorname{Ob}\underline{\mathcal{V}\text{-}\mathsf{Quiv}}(\mathfrak{a},\mathfrak{b})=\mathcal{S}\mathrm{et}(\operatorname{Ob}\mathfrak{a},\operatorname{Ob}\mathfrak{b})$, and for any pair of maps $f,g:\operatorname{Ob}\mathfrak{a}\to\operatorname{Ob}\mathfrak{b}$

$$\underline{\mathcal{V}\text{-}\mathsf{Quiv}}(\mathfrak{a},\mathfrak{b})(f,g)=\prod_{X,Y\in\operatorname{Ob}\mathfrak{a}}\underline{\mathcal{V}}(\mathfrak{a}(X,Y),\mathfrak{b}(fX,gY)).$$

The evaluation morphism $\mathrm{ev}:\mathfrak{a}\boxtimes\underline{\mathcal{V}\text{-}\mathsf{Quiv}}(\mathfrak{a},\mathfrak{b})\to\mathfrak{b}$ (the adjunct of $1_{\underline{\mathcal{V}\text{-}\mathsf{Quiv}}(\mathfrak{a},\mathfrak{b})}$) is given by

$$(X,f)\mapsto fX,$$
$$\mathfrak{a}(X,Y)\otimes\prod_{X',Y'\in\operatorname{Ob}\mathfrak{a}}\underline{\mathcal{V}}(\mathfrak{a}(X',Y'),\mathfrak{b}(fX',gY'))\xrightarrow{1\otimes\mathrm{pr}_{X,Y}}$$
$$\mathfrak{a}(X,Y)\otimes\underline{\mathcal{V}}(\mathfrak{a}(X,Y),\mathfrak{b}(fX,gY))\xrightarrow{\mathrm{ev}}\mathfrak{b}(fX,gY).$$

By definition there is a functor $\operatorname{Ob}:\mathcal{V}$-$\mathsf{Quiv}\to\mathcal{S}\mathrm{et}$, $\mathfrak{a}\mapsto\operatorname{Ob}\mathfrak{a}$. Consider the fiber $\mathcal{V}$-Quiv_S of this functor over a set S, that is,

$$\operatorname{Ob}\mathcal{V}\text{-}\mathsf{Quiv}_S=\{\mathfrak{a}\in\mathcal{V}\text{-}\mathsf{Quiv}\mid\operatorname{Ob}\mathfrak{a}=S\},$$
$$\operatorname{Mor}\mathcal{V}\text{-}\mathsf{Quiv}_S=\{f\in\operatorname{Mor}\mathcal{V}\text{-}\mathsf{Quiv}\mid\operatorname{Ob}f=\mathrm{id}_S\}.$$

Since $\mathcal{V}$ is abelian, so is $\mathcal{V}$-Quiv_S being isomorphic to $\mathcal{V}^{S\times S}$.

1.3. Definition. *The category $\mathcal{V}$-Quiv_S is monoidal with the tensor product $\otimes$*

$$(\mathfrak{a}\otimes\mathfrak{b})(X,Z)=\coprod_{Y\in S}\mathfrak{a}(X,Y)\otimes\mathfrak{b}(Y,Z),$$

and the unit object $\mathbb{1}S$.

For an arbitrary $\mathcal{V}$-quiver $\mathfrak{a}$ we denote $T^n\mathfrak{a}=\mathfrak{a}^{\otimes n}\in\mathcal{V}\text{-}\mathsf{Quiv}_{\operatorname{Ob}\mathfrak{a}}$, $n\geqslant 0$, $T^0\mathfrak{a}=\mathbb{1}\operatorname{Ob}\mathfrak{a}=\mathbb{1}\mathfrak{a}$. We use $\mathbb{1}\mathfrak{a}$ as a shorthand for $\mathbb{1}\operatorname{Ob}\mathfrak{a}$ and $\mathbb{1}f$ as a shorthand for $\mathbb{1}\operatorname{Ob}f$, where f is a morphism of quivers.

Let $\mathfrak{a}_1,\ldots,\mathfrak{a}_n$, $\mathfrak{b}_1,\ldots,\mathfrak{b}_n$ be $\mathcal{V}$-quivers with $\operatorname{Ob}\mathfrak{a}_1=\cdots=\operatorname{Ob}\mathfrak{a}_n=S$, $\operatorname{Ob}\mathfrak{b}_1=\cdots=\operatorname{Ob}\mathfrak{b}_n=Q$. Let $f_i:\mathfrak{a}_i\to\mathfrak{b}_i$, $1\leqslant i\leqslant n$, be morphisms of quivers such that $\operatorname{Ob}f_i=f:S\to Q$. Then the following morphism is well-defined

$$f_1\otimes\cdots\otimes f_n:\mathfrak{a}_1\otimes\cdots\otimes\mathfrak{a}_n\to\mathfrak{b}_1\otimes\cdots\otimes\mathfrak{b}_n,$$
$$\operatorname{Ob}f_1\otimes\cdots\otimes f_n=f,$$
$$\begin{aligned}f_1\otimes\cdots\otimes f_n=\Big[\coprod_{X_1,\ldots,X_{n-1}\in S}&\mathfrak{a}_1(X_0,X_1)\otimes\cdots\otimes\mathfrak{a}_n(X_{n-1},X_n)\xrightarrow{f_1\otimes\cdots\otimes f_n}\\ \coprod_{X_1,\ldots,X_{n-1}\in S}&\mathfrak{b}_1(fX_0,fX_1)\otimes\cdots\otimes\mathfrak{b}_n(fX_{n-1},fX_n)\xrightarrow{(\mathrm{in}_{fX_1,\ldots,fX_{n-1}})_{X_1,\ldots,X_{n-1}\in S}}\\ &\coprod_{Y_1,\ldots,Y_{n-1}\in Q}\mathfrak{b}_1(fX_0,Y_1)\otimes\cdots\otimes\mathfrak{b}_n(Y_{n-1},fX_n)\Big].\end{aligned}\tag{1.1}$$

In the case $n=0$ (when a map $f:S\to Q$ is given) $f_1\otimes\cdots\otimes f_n$ is the morphism $\mathbb{1}f:\mathbb{1}S\to\mathbb{1}Q$.

1.4. Definition. *A cocategory $\mathfrak{c}$ is a coalgebra in the monoidal category $\mathcal{V}$-Quiv_S. Of course, $S=\operatorname{Ob}\mathfrak{c}$. In other words, $\mathfrak{c}$ is a $\mathcal{V}$-quiver equipped with a coassociative comultiplication $\Delta:\mathfrak{c}\to\mathfrak{c}\otimes\mathfrak{c}$ and the counit $\varepsilon:\mathfrak{c}\to\mathbb{1}\mathfrak{c}$ which satisfies the usual counitality equations. Morphisms of cocategories (cofunctors) $f:\mathfrak{b}\to\mathfrak{c}$ are morphisms of $\mathcal{V}$-quivers compatible with the comultiplication and the counit in the sense that*

$$\begin{array}{ccc}\mathfrak{b}&\xrightarrow{f}&\mathfrak{c}\\ {\scriptstyle\Delta}\downarrow&=&\downarrow{\scriptstyle\Delta}\\ \mathfrak{b}\otimes\mathfrak{b}&\xrightarrow{f\otimes f}&\mathfrak{c}\otimes\mathfrak{c}\end{array}\quad,\qquad\begin{array}{ccc}\mathfrak{b}&\xrightarrow{f}&\mathfrak{c}\\ {\scriptstyle\varepsilon}\downarrow&=&\downarrow{\scriptstyle\varepsilon}\\ \mathbb{1}\mathfrak{b}&\xrightarrow{\mathbb{1}f}&\mathbb{1}\mathfrak{c}\end{array}\quad.\tag{1.2}$$

The category of cocategories is denoted coCat.

1.5. Example. For any set S the $\mathcal{V}$-quiver $\mathbb{1}S$ is a cocategory with the identity morphism $\mathrm{id}_{\mathbb{1}S}$ as ε and the isomorphism $\Delta:\mathbb{1}S\to\mathbb{1}S\otimes\mathbb{1}S$, coming from the canonical isomorphism $\mathbb{1}\cong\mathbb{1}\otimes\mathbb{1}$.

1.6. Definition. *An augmented cocategory* $\mathfrak{c}$ *is a coalgebra morphism* $\eta : \mathbb{1}\mathfrak{c} \to \mathfrak{c}$ *in* $\mathcal{V}$-$\mathsf{Quiv}_{\mathrm{Ob}\,\mathfrak{c}}$. *Morphisms of augmented cocategories* $f : \mathfrak{b} \to \mathfrak{c}$ *are morphisms of cocategories compatible with the augmentation, that is,*

$$\begin{array}{ccc} \mathbb{1}\mathfrak{b} & \xrightarrow{\mathbb{1}f} & \mathbb{1}\mathfrak{c} \\ {\scriptstyle\eta}\downarrow & = & \downarrow{\scriptstyle\eta} \\ \mathfrak{b} & \xrightarrow{f} & \mathfrak{c} \end{array} \tag{1.3}$$

The category of augmented cocategories is denoted acCat.

Notice that $\mathrm{Ob}\,\eta = \mathrm{id}_{\mathrm{Ob}\,\mathfrak{c}}$ for an augmented cocategory $\mathfrak{c}$. It follows from (1.2) that $\eta \cdot \varepsilon = 1_{\mathfrak{c}}$.

Recall that the category $\mathcal{V} = \mathbf{gr}$ is idempotent complete. An augmented cocategory $\mathfrak{c}$ splits into a direct sum $\mathfrak{c} = \mathbb{1}\mathfrak{c} \oplus \bar{\mathfrak{c}}$ in $\mathcal{V}$-$\mathsf{Quiv}_{\mathrm{Ob}\,\mathfrak{c}}$ so that ε becomes pr_1 and η becomes in_1. A non-counital comultiplication, induced on $\bar{\mathfrak{c}}$,

$$\bar{\Delta} = \Delta - \eta \otimes 1 - 1 \otimes \eta : \bar{\mathfrak{c}} \to \bar{\mathfrak{c}} \otimes \bar{\mathfrak{c}} \in \mathcal{V}\text{-}\mathsf{Quiv}_{\mathrm{Ob}\,\mathfrak{c}} \tag{1.4}$$

is coassociative. In fact, $\mathrm{pr}_2 : \mathfrak{c} \to \bar{\mathfrak{c}}$ identifies with the canonical projection $\pi : \mathfrak{c} \to \mathrm{Coker}\,\eta = \mathfrak{c}/\operatorname{Im}\eta$ and $\bar{\Delta}$ can be found from

$$\begin{array}{ccc} \mathfrak{c} & \xrightarrow{\Delta} & \mathfrak{c} \otimes \mathfrak{c} \\ {\scriptstyle\pi}\downarrow & = & \downarrow{\scriptstyle\pi\otimes\pi} \\ \bar{\mathfrak{c}} & \xrightarrow{\bar{\Delta}} & \bar{\mathfrak{c}} \otimes \bar{\mathfrak{c}} \end{array} \tag{1.5}$$

1.7. Definition. *An augmented cocategory* $\mathfrak{c}$ *is called* conilpotent *when* $(\bar{\mathfrak{c}}, \bar{\Delta})$ *is conilpotent, that is,*

$$\bigcup_{n>1} \mathrm{Ker}(\bar{\Delta}^{(n)} : \bar{\mathfrak{c}} \to \bar{\mathfrak{c}}^{\otimes n}) = \bar{\mathfrak{c}}.$$

The full subcategory of acCat *whose objects are conilpotent cocategories is denoted* ncCat.

1.8. Example. For an arbitrary $\mathcal{V}$-quiver $\mathfrak{a}$ there is the tensor quiver $T\mathfrak{a} = \coprod_{n\geqslant 0} T^n\mathfrak{a} = \coprod_{n\geqslant 0} \mathfrak{a}^{\otimes n} \equiv \oplus_{n\geqslant 0}\mathfrak{a}^{\otimes n}$. Define comultiplication $\Delta : T\mathfrak{a} \to T\mathfrak{a} \otimes T\mathfrak{a}$ as the sum of canonical isomorphisms $\mathfrak{a}^{\otimes n} \to \mathfrak{a}^{\otimes k} \otimes \mathfrak{a}^{\otimes l}$, $k + l = n$, $k, l \geqslant 0$. On elements

$$\Delta(h_1 \otimes h_2 \otimes \cdots \otimes h_n) = \sum_{k=0}^{n} h_1 \otimes \cdots \otimes h_k \bigotimes h_{k+1} \otimes \cdots \otimes h_n$$

is the cut comultiplication. The counit is $\varepsilon = \underline{\mathrm{pr}}_0 : T\mathfrak{a} \to T^0\mathfrak{a} = \mathbb{1}\mathfrak{a}$, the augmentation is $\eta = \mathrm{in}_0 : T^0\mathfrak{a} \to T\mathfrak{a}$. The direct summand $\overline{T\mathfrak{a}} = T^{>0}\mathfrak{a} = \oplus_{n>0} T^n\mathfrak{a}$ is equipped with the reduced comultiplication $\bar{\Delta} : T^{>0}\mathfrak{a} \to T^{>0}\mathfrak{a} \otimes T^{>0}\mathfrak{a}$ which is the sum of canonical isomorphisms $\mathfrak{a}^{\otimes n} \to \mathfrak{a}^{\otimes k} \otimes \mathfrak{a}^{\otimes l}$, $k + l = n$, $k, l > 0$. On elements

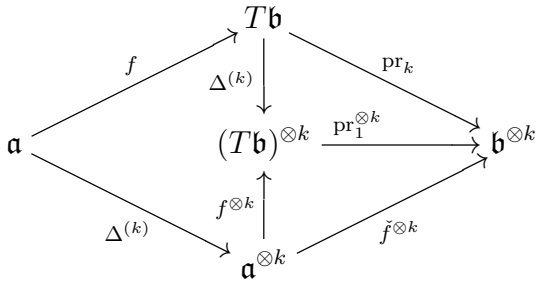
$$\bar{\Delta}(h_1 \otimes h_2 \otimes \cdots \otimes h_n) = \sum_{k=1}^{n-1} h_1 \otimes \cdots \otimes h_k \bigotimes h_{k+1} \otimes \cdots \otimes h_n.$$

Since it is conilpotent, the augmented cocategory $T\mathfrak{a}$ is conilpotent. It is called the cofree conilpotent cocategory. The reasons are clear from the following

1.9. Proposition. *The tensor cocategory construction extends to a functor* $T : \mathcal{V}\text{-}\mathsf{Quiv} \to$ ncCat*, representing the left hand side of a natural bijection*

$$\{\phi : \mathfrak{a} \to \mathfrak{b} \in \mathcal{V}\text{-}\mathsf{Quiv} \mid \eta \cdot \phi = 0\} \cong \mathrm{ncCat}(\mathfrak{a}, T\mathfrak{b}).$$

Proof. For any $g : \mathfrak{a} \to \mathfrak{b} \in \mathcal{V}\text{-}\mathsf{Quiv}$ the morphisms $T^n g = g^{\otimes n}$ constructed in (1.1) form Tg and extend T to a functor. Any morphism $f : \mathfrak{a} \to T\mathfrak{b} \in \mathrm{ncCat}$ is uniquely determined by the composition $\check{f} = f \cdot \mathrm{pr}_1 : \mathfrak{a} \to \mathfrak{b} \in \mathcal{V}\text{-}\mathsf{Quiv}$ such that $\eta \cdot \check{f} = 0$, as the following commutative diagram shows



Namely, the following expression makes sense in notation $x_{(1)} \otimes \cdots \otimes x_{(k)} \equiv \Delta^{(k)}(x)$

$$f(x) = \sum_{k \geqslant 0} \check{f}(x_{(1)}) \otimes \cdots \otimes \check{f}(x_{(k)}) = (x)\varepsilon \cdot (\mathbb{1} f) \cdot \mathrm{in}_0 + \sum_{k \geqslant 1} \check{f}(x_{(1)}) \otimes \cdots \otimes \check{f}(x_{(k)}),$$

since $\bar{\Delta}^{(k)}(x) = 0$ for large k and $\eta \cdot \check{f} = 0$. ∎

The category coCat is symmetric monoidal with the tensor product $\boxtimes$ given by

$$\Big(\mathfrak{a} \boxtimes \mathfrak{b}, \Delta = \big(\mathfrak{a} \boxtimes \mathfrak{b} \xrightarrow{\Delta\boxtimes\Delta} (\mathfrak{a} \otimes \mathfrak{a}) \boxtimes (\mathfrak{b} \otimes \mathfrak{b}) \xrightarrow{\oplus 1 \otimes c \otimes 1} (\mathfrak{a} \boxtimes \mathfrak{b}) \otimes (\mathfrak{a} \boxtimes \mathfrak{b})\big),$$
$$\varepsilon = \big(\mathfrak{a} \boxtimes \mathfrak{b} \xrightarrow{\varepsilon\boxtimes\varepsilon} \mathbb{1}\mathfrak{a} \boxtimes \mathbb{1}\mathfrak{b} \cong \mathbb{1}(\mathrm{Ob}\,\mathfrak{a} \times \mathrm{Ob}\,\mathfrak{b}) = \mathbb{1}(\mathfrak{a} \boxtimes \mathfrak{b})\big)\Big).$$

The isomorphism $\tau_{(23)} = \oplus 1 \otimes \tau \otimes 1 : (\mathfrak{a} \otimes \mathfrak{a}) \boxtimes (\mathfrak{b} \otimes \mathfrak{b}) \to (\mathfrak{a} \boxtimes \mathfrak{b}) \otimes (\mathfrak{a} \boxtimes \mathfrak{b})$ (the middle four interchange) is the direct sum of isomorphisms

$$1 \otimes \tau \otimes 1 : \mathfrak{a}(X,Y) \otimes \mathfrak{a}(Y,Z) \otimes \mathfrak{b}(U,V) \otimes \mathfrak{b}(V,W) \to \mathfrak{a}(X,Y) \otimes \mathfrak{b}(U,V) \otimes \mathfrak{a}(Y,Z) \otimes \mathfrak{b}(V,W).$$

Hence the category acCat is symmetric monoidal. The augmentation for the tensor product $\mathfrak{a} \boxtimes \mathfrak{b}$ of augmented cocategories $\mathfrak{a}$, $\mathfrak{b}$ is

$$\eta = \big(\mathbb{1}(\mathfrak{a} \boxtimes \mathfrak{b}) = \mathbb{1}(\mathrm{Ob}\,\mathfrak{a} \times \mathrm{Ob}\,\mathfrak{b}) \cong \mathbb{1}\mathfrak{a} \boxtimes \mathbb{1}\mathfrak{b} \xrightarrow{\eta\boxtimes\eta} \mathfrak{a} \boxtimes \mathfrak{b}\big).$$

1.10. Proposition. *The category* ncCat *is a full monoidal subcategory of* acCat*.*

PROOF. Given two conilpotent cocategories $\mathfrak{c}$ and $\mathfrak{d}$, let us prove that $\mathfrak{c}\boxtimes\mathfrak{d}$ is conilpotent. The canonical projections $\pi:\mathfrak{c}\to\bar{\mathfrak{c}}$ and $\pi:\mathfrak{d}\to\bar{\mathfrak{d}}$ allow to write $\pi:\mathfrak{c}\boxtimes\mathfrak{d}\to\overline{\mathfrak{c}\boxtimes\mathfrak{d}}$ as

$$(\pi\boxtimes\pi,\pi\boxtimes\varepsilon,\varepsilon\boxtimes\pi):\mathfrak{c}\boxtimes\mathfrak{d}\to\bar{\mathfrak{c}}\boxtimes\bar{\mathfrak{d}}\oplus\bar{\mathfrak{c}}\boxtimes\mathbb{1}\mathfrak{d}\oplus\mathbb{1}\mathfrak{c}\boxtimes\bar{\mathfrak{d}}.$$

Diagram (1.5) implies

$$\begin{array}{ccc}\mathfrak{c} & \xrightarrow{\Delta^{(l)}} & \mathfrak{c}^{\otimes l}\\ {\scriptstyle\pi}\downarrow & = & \downarrow{\scriptstyle\pi^{\otimes l}}\\ \bar{\mathfrak{c}} & \xrightarrow{\bar\Delta^{(l)}} & \bar{\mathfrak{c}}^{\otimes l}\end{array}$$

and similarly for $\mathfrak{d}$ and $\mathfrak{c}\boxtimes\mathfrak{d}$. Hence,

$$\begin{array}{ccccc}
 & & \overset{\Delta^{(l)}\boxtimes\Delta^{(l)}}{\overbrace{\qquad\qquad\qquad\qquad\qquad\qquad}} & & \\
 & & = & & \\
\mathfrak{c}\boxtimes\mathfrak{d} & \xrightarrow{\Delta^{(l)}} & (\mathfrak{c}\boxtimes\mathfrak{d})^{\otimes l} & \xrightarrow[\cong]{\text{unshuffle}} & \mathfrak{c}^{\otimes l}\boxtimes\mathfrak{d}^{\otimes l}\\
\downarrow{\scriptstyle\begin{pmatrix}\pi\boxtimes\pi\\ \pi\boxtimes\varepsilon\\ \varepsilon\boxtimes\pi\end{pmatrix}} & = & \downarrow{\scriptstyle\begin{pmatrix}\pi\boxtimes\pi\\ \pi\boxtimes\varepsilon\\ \varepsilon\boxtimes\pi\end{pmatrix}^{\otimes l}} & = & \downarrow{\scriptstyle(\nu(i_1)\otimes\cdots\otimes\nu(i_l))\boxtimes\nu(j_1)\otimes\cdots\otimes\nu(j_l))_{i,j}}\\
\begin{matrix}\bar{\mathfrak{c}}\boxtimes\bar{\mathfrak{d}}\\ \oplus\\ \bar{\mathfrak{c}}\boxtimes\mathbb{1}\mathfrak{d}\\ \oplus\\ \mathbb{1}\mathfrak{c}\boxtimes\bar{\mathfrak{d}}\end{matrix} & \xrightarrow{\bar\Delta^{(l)}} & \begin{pmatrix}\bar{\mathfrak{c}}\boxtimes\bar{\mathfrak{d}}\\ \oplus\\ \bar{\mathfrak{c}}\boxtimes\mathbb{1}\mathfrak{d}\\ \oplus\\ \mathbb{1}\mathfrak{c}\boxtimes\bar{\mathfrak{d}}\end{pmatrix} & \xrightarrow[\cong]{\text{unshuffle}} & \bigoplus_{i_k+j_k>0}^{i_k,j_k\in\{0,1\}}\bar{\mathfrak{c}}^{\otimes i_1}\otimes\cdots\otimes\bar{\mathfrak{c}}^{\otimes i_l}\boxtimes\bar{\mathfrak{d}}^{\otimes j_1}\otimes\cdots\otimes\bar{\mathfrak{d}}^{\otimes j_l}
\end{array}$$

where $\nu(0)=\varepsilon$ and $\nu(1)=\pi$. Denote by $p=\#\{k\mid i_k=1\}$ and $q=\#\{k\mid j_k=1\}$ certain cardinalities. The canonical isomorphism of the summand in the bottom right corner with $\bar{\mathfrak{c}}^{\otimes p}\boxtimes\bar{\mathfrak{d}}^{\otimes q}$ satisfies

$$\Delta^{(l)}\cdot(\nu(i_1)\otimes\cdots\otimes\nu(i_l))\cdot\mathrm{iso}=\Delta^{(p)}\cdot\pi^{\otimes p}=\pi\cdot\bar\Delta^{(p)}:\mathfrak{c}\to\bar{\mathfrak{c}}^{\otimes p},$$
$$\Delta^{(l)}\cdot(\nu(j_1)\otimes\cdots\otimes\nu(j_l))\cdot\mathrm{iso}=\Delta^{(q)}\cdot\pi^{\otimes q}=\pi\cdot\bar\Delta^{(q)}:\mathfrak{d}\to\bar{\mathfrak{d}}^{\otimes q}.$$

Clearly, $p+q\geqslant l$. Therefore, if $c\in\bar{\mathfrak{c}}$, $\bar\Delta^{(n)}c=0$ and $d\in\bar{\mathfrak{d}}$, $\bar\Delta^{(m)}d=0$, then the lower row applied to $c\boxtimes d$ vanishes for $l=n+m-1$. If $c\in\bar{\mathfrak{c}}$, $\bar\Delta^{(n)}c=0$ and $U\in\mathrm{Ob}\,\mathfrak{d}$, then the lower row applied to $c\boxtimes\eta_U$ ends up only in the summand with $i_1=\cdots=i_l=1$, $j_1=\cdots=j_l=0$. Hence, $\bar\Delta^{(l)}(c\boxtimes\eta_U)=0$ for $l=n$. Similarly, if $X\in\mathrm{Ob}\,\mathfrak{c}$ and $d\in\bar{\mathfrak{d}}$, $\bar\Delta^{(m)}d=0$, then $\bar\Delta^{(m)}(\eta_X\boxtimes d)=0$. ∎

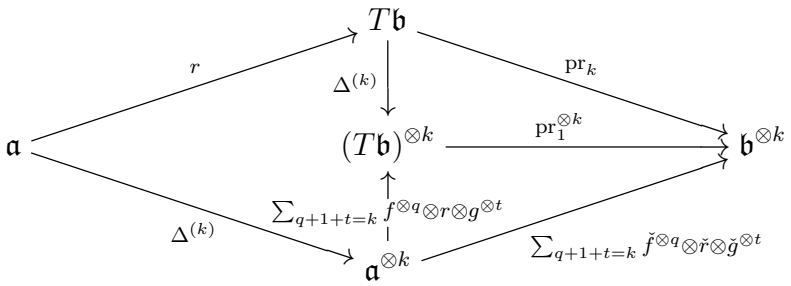
Being a full monoidal subcategory of acCat the category ncCat is symmetric.

1.11. DEFINITION. *Let $f,g:\mathfrak{a}\to\mathfrak{b}\in\mathrm{coCat}$. An (f,g)-coderivation $r:f\to g:\mathfrak{a}\to\mathfrak{b}$ of degree d is a collection of morphisms $r:\mathfrak{a}(X,Y)\to\mathfrak{b}(fX,gY)$ of degree d, which satisfies the equation $r\cdot\Delta=\Delta\cdot(f\otimes r+r\otimes g)$.*

The maps $f\otimes r,r\otimes g:\mathfrak{a}\otimes\mathfrak{a}\to\mathfrak{b}\otimes\mathfrak{b}$ are defined similarly to (1.1).

1.12. Proposition. *Let $f, g : \mathfrak{a} \to T\mathfrak{b} \in \mathrm{ncCat}$. (f,g)-coderivations $r : f \to g : \mathfrak{a} \to T\mathfrak{b}$ of degree d are in bijection with the collections of morphisms $\check{r} = r \cdot \mathrm{pr}_1 : \mathfrak{a}(X,Y) \to \mathfrak{b}(fX, gY)$ of degree d.*

Proof. The commutative diagram



shows that r is given by the formula

$$r = \sum_{k \geqslant 1} \Delta^{(k)} \cdot \sum_{q+1+t=k} \check{f}^{\otimes q} \otimes \check{r} \otimes \check{g}^{\otimes t} = \Delta^{(3)} \cdot (f \otimes \check{r} \otimes g). \tag{1.6}$$

The first expression makes sense, since in each term of decomposition of $\Delta^{(k)}$ there are factors of $\bar{\Delta}^{(n)}$ and $k-n$ unit morphisms (elements $\eta(1)$). The maps $\check{f}$ and $\check{g}$ vanish on the latter, hence, if $\bar{\Delta}^{(k-1)}(x) = 0$, then k-th term of (1.6) vanishes on x. The second expression obviously makes sense. ∎

Let $\mathfrak{a}, \mathfrak{b} \in \mathrm{acCat}$. The coderivation quiver $\mathrm{Coder}(\mathfrak{a}, \mathfrak{b})$ has augmentation preserving cofunctors $f : \mathfrak{a} \to \mathfrak{b}$ as objects and the d-th component of the graded Λ-module $\mathrm{Coder}(\mathfrak{a}, \mathfrak{b})(f,g)$ consists of coderivations $r : f \to g : \mathfrak{a} \to \mathfrak{b}$ of degree d. Notice that in [Lyubashenko, 2003] the notation $\mathrm{Coder}(A,B)$ was used as a shorthand for $\mathrm{Coder}(TA, TB)$ for $\mathcal{V}$-quivers A and B.

Let $\phi : \mathfrak{a} \boxtimes \mathfrak{c} \to \mathfrak{b}$ be a cocategory homomorphism of degree 0. By definition the homomorphism ϕ satisfies the equation

$$\begin{array}{ccccc} \mathfrak{a} \boxtimes \mathfrak{c} & & \xrightarrow{\phi} & & \mathfrak{b} \xrightarrow{\Delta} \mathfrak{b} \otimes \mathfrak{b} \\ \downarrow{\scriptstyle \Delta\boxtimes\Delta} & & = & & \uparrow{\scriptstyle \phi\otimes\phi} \\ (\mathfrak{a} \otimes \mathfrak{a}) \boxtimes (\mathfrak{c} \otimes \mathfrak{c}) & & \xrightarrow{1\otimes c\otimes 1} & & (\mathfrak{a} \boxtimes \mathfrak{c}) \otimes (\mathfrak{a} \boxtimes \mathfrak{c}) \end{array}$$

Let $c \in \mathfrak{c}^n$ (in the next several paragraphs c does not mean the symmetry). Introduce $c\chi : \mathfrak{a} \to \mathfrak{b} \in \underline{\mathcal{V}\text{-Quiv}}^n$ by the formula $a(c\chi) = (a \boxtimes c)\phi$. Then the above equation is equivalent to

$$a(c\chi)\Delta = a\Delta(c_{(1)}\chi \otimes c_{(2)}\chi). \tag{1.7}$$

Another equation satisfied by ϕ is counitality: $(a \boxtimes c)\phi\varepsilon \equiv a(c\chi)\varepsilon = (a\varepsilon)(c\varepsilon)$.

Assume that $\mathfrak{a}, \mathfrak{b} \in \mathrm{coCat}$ and $\mathfrak{c} \in \mathrm{acCat}$. Given a cofunctor $\phi : \mathfrak{a} \boxtimes \mathfrak{c} \to \mathfrak{b}$ and an object $C \in \mathfrak{c}$ there is a cofunctor $(\_ \boxtimes C)\phi : \mathfrak{a} \to \mathfrak{b}$, which acts on objects as $(\_ \boxtimes C)\phi : \mathrm{Ob}\,\mathfrak{a} \to$

$\operatorname{Ob}\mathfrak{b}$, $A \mapsto (A\boxtimes C)\phi$ and on morphisms as $(\_\boxtimes 1_C\eta)\phi : \mathfrak{a}(A', A'') \to \mathfrak{b}((A'\boxtimes C)\phi, (A''\boxtimes C)\phi)$, $a \mapsto (a \boxtimes 1_C\eta)\phi$, where $(1_C)\eta \in \mathfrak{c}(C, C)$, $1_C = 1 \in (\mathbb{1}\mathfrak{c})(C, C) = \Lambda$. If, furthermore, $\mathfrak{a}$ and $\mathfrak{b}$ are augmented and ϕ preserves augmentation, so does $(\_ \boxtimes C)\phi$ for any $C \in \operatorname{Ob}\mathfrak{c}$.

Assume that $c \in \mathcal{C}(f, g)^d$ satisfies $c\Delta = 1_f\eta \otimes c + c \otimes 1_g\eta$. Then the collection $\xi : \mathfrak{a}(A', A'') \to \mathfrak{b}((A' \boxtimes f)\phi, (A'' \boxtimes g)\phi)$, $a \mapsto (a\boxtimes c)\phi$, is a $((\_\boxtimes f)\phi, (\_\boxtimes g)\phi)$-coderivation of degree d.

1.13. Evaluation. Let $\mathfrak{a}$ be a conilpotent cocategory and let $\mathfrak{b}$ be a $\mathcal{V}$-quiver. Define the evaluation cofunctor $\phi = \mathrm{ev} : \mathfrak{a}\boxtimes T\operatorname{Coder}(\mathfrak{a}, T\mathfrak{b}) \to T\mathfrak{b}$ on objects as $\mathrm{ev}(A\boxtimes f) = fA$, and on morphisms by the corresponding χ. Let $f^0, f^1, \dots, f^n : \mathfrak{a} \to T\mathfrak{b}$ be cofunctors, and let $r^1, \dots, r^n$ be coderivations of certain degrees as in $f^0 \xrightarrow{r^1} f^1 \xrightarrow{r^2} \dots f^{n-1} \xrightarrow{r^n} f^n : \mathfrak{a} \to T\mathfrak{b}$, $n \geqslant 0$. Then $c = r^1 \otimes \dots \otimes r^n \in T^n \operatorname{Coder}(\mathfrak{a}, T\mathfrak{b})(f^0, f^n)$. Define $(a \boxtimes c)\,\mathrm{ev} = a.(c\chi)$ as

$$(a\boxtimes(r^1\otimes\dots\otimes r^n))\,\mathrm{ev} = (a\Delta^{(2n+1)})(f^0\otimes\check r^1\otimes f^1\otimes\check r^2\otimes\dots\otimes f^{n-1}\otimes\check r^n\otimes f^n)\mu_{T\mathfrak{b}}^{(2n+1)}. \quad (1.8)$$

The right hand side belongs to $(T\mathfrak{b})^{\otimes(2n+1)}\mu_{T\mathfrak{b}}^{(2n+1)}$ and is mapped by multiplication $\mu_{T\mathfrak{b}}^{(2n+1)}$ into $T\mathfrak{b}$. In particular, for $n = 1$ we have $(a \boxtimes r^1)\,\mathrm{ev} = (a)r^1$ due to (1.6). In order to see that ev is a cofunctor we verify (1.7):

$$\begin{aligned}
a.(c\chi)\Delta &= a\Delta^{(2n+1)}(f^0\otimes\check r^1\otimes f^1\otimes\check r^2\otimes\dots\otimes f^{n-1}\otimes\check r^n\otimes f^n)\mu_{T\mathfrak{b}}^{(2n+1)}\Delta\\
&= a\Delta^{(2n+1)}\sum_{m=0}^{n}(f^0\otimes\check r^1\otimes\dots\otimes f^{m-1}\otimes\check r^m\otimes f^m\Delta\otimes\check r^{m+1}\otimes f^{m+1}\otimes\dots\otimes\check r^n\otimes f^n)\\
&\qquad\qquad(\mu_{T\mathfrak{b}}^{(2m+1)}\bigotimes\mu_{T\mathfrak{b}}^{(2n-2m+1)})\\
&= a\Delta^{(2n+1)}\sum_{m=0}^{n}\big(f^0\otimes\check r^1\otimes\dots\otimes f^{m-1}\otimes\check r^m\otimes\Delta(f^m\bigotimes f^m)\\
&\qquad\qquad\otimes\check r^{m+1}\otimes f^{m+1}\otimes\dots\otimes\check r^n\otimes f^n\big)(\mu_{T\mathfrak{b}}^{(2m+1)}\bigotimes\mu_{T\mathfrak{b}}^{(2n-2m+1)})\\
&= a\Delta\sum_{m=0}^{n}(\Delta^{(2m+1)}\bigotimes\Delta^{(2n-2m+1)})\big[(f^0\otimes\check r^1\otimes\dots\otimes f^{m-1}\otimes\check r^m\otimes f^m)\mu_{T\mathfrak{b}}^{(2m+1)}\\
&\qquad\qquad\bigotimes(f^m\otimes\check r^{m+1}\otimes f^{m+1}\otimes\dots\otimes\check r^n\otimes f^n)\mu_{T\mathfrak{b}}^{(2n-2m+1)}\big]\\
&= a\Delta(c_{(1)}\chi\bigotimes c_{(2)}\chi).
\end{aligned}$$

Here we have used $\otimes$ for the product in the tensor quiver $T\mathfrak{b}$ and $\bigotimes$ for $T\mathfrak{b}\bigotimes T\mathfrak{b}$. The counitality equation for ev has to be proven for $n = 0$, where it reduces to counitality of f^0. The cofunctor ev preserves the augmentation, since all $f \in \operatorname{Ob}\operatorname{Coder}(\mathfrak{a}, T\mathfrak{b})$ do.

The following is a version of Proposition 3.4 of [Lyubashenko, 2003].

1.14. Proposition. *For $\mathfrak{a} \in \mathrm{ncCat}$, $\mathfrak{b}, \mathfrak{c}^1, \dots \mathfrak{c}^q \in \mathcal{V}$-Quiv with notation $\mathfrak{c} = T\mathfrak{c}^1\boxtimes\dots\boxtimes T\mathfrak{c}^q$ the map*

$$\begin{aligned}
\mathrm{ncCat}(\mathfrak{c}, T\operatorname{Coder}(\mathfrak{a}, T\mathfrak{b})) &\longrightarrow \mathrm{ncCat}(\mathfrak{a}\boxtimes\mathfrak{c}, T\mathfrak{b}),\\
\psi &\longmapsto \big(\mathfrak{a}\boxtimes\mathfrak{c}\xrightarrow{\mathfrak{a}\boxtimes\psi}\mathfrak{a}\boxtimes T\operatorname{Coder}(\mathfrak{a}, T\mathfrak{b})\xrightarrow{\mathrm{ev}}T\mathfrak{b}\big)
\end{aligned}$$

is a bijection.

We give a new

PROOF. An augmentation preserving cofunctor $\psi : \mathfrak{c} \to T\operatorname{Coder}(\mathfrak{a}, T\mathfrak{b})$ is described by an arbitrary quiver map $\check{\psi} = \psi \cdot \mathrm{pr}_1 : \mathfrak{c} \to \operatorname{Coder}(\mathfrak{a}, T\mathfrak{b}) \in \Lambda\text{-mod}^{\mathrm{c}}_{\mathbb{L}}\text{-}\mathsf{Quiv}$ such that $\eta \cdot \check{\psi} = 0$. Let $\phi : \mathfrak{a} \boxtimes \mathfrak{c} \to T\mathfrak{b}$ be an augmentation preserving cofunctor. We have to satisfy the equation

$$\sum_{k \geqslant 0} (a \boxtimes c\Delta^{(k)}\check{\psi}^{\otimes k}) \operatorname{ev} = (a \boxtimes c)\phi, \qquad a \in \mathfrak{a}^\bullet,\ c \in \mathfrak{c}^\bullet.$$

It suffices to consider two cases. In the first one $c = \eta(1_g)$ for some $g \in \operatorname{Ob}\mathfrak{c}$. Then the equation takes the form $(a)(g\psi) = (a \boxtimes c)\phi$ which defines the cofunctor $g\psi \in \operatorname{ncCat}(\mathfrak{a}, T\mathfrak{b})$ in the left hand side.

In the second case $c \in \bar{\mathfrak{c}}^d$, $d \in \mathbb{Z}$, the equation takes the form

$$(a)(c)\check{\psi} + \sum_{k \geqslant 2} (a \boxtimes c\Delta^{(k)}\check{\psi}^{\otimes k}) \operatorname{ev} = (a \boxtimes c)\phi, \qquad a \in \mathfrak{a}^\bullet,\ c \in \bar{\mathfrak{c}}^\bullet.$$

Since $\eta \cdot \check{\psi} = 0$ the comultiplication Δ can be replaced with $\bar{\Delta}$. The structure of $\mathfrak{c} = T\mathfrak{c}^1 \boxtimes \cdots \boxtimes T\mathfrak{c}^q$ is such that the component $\psi_{i_1,\ldots,i_q}$ in the left hand side of

$$(a)(c)\check{\psi} = (a \boxtimes c)\phi - \sum_{k \geqslant 2} (a \boxtimes c\bar{\Delta}^{(k)}\check{\psi}^{\otimes k}) \operatorname{ev}, \qquad a \in \mathfrak{a}^\bullet,\ c \in \bar{\mathfrak{c}}^\bullet. \tag{1.9}$$

is expressed via the components $\psi_{j_1,\ldots,j_q}$ with smaller indices $(j_1, \ldots, j_q)$ in the product poset $\mathbb{N}^q$. For $c \in \bar{\mathfrak{c}}(X, Y)^d$, $X = (X_1, \ldots, X_q)$, $Y = (Y_1, \ldots, Y_q)$, $X_i, Y_i \in \operatorname{Ob}\mathfrak{c}^i$, find $n \geqslant 0$ such that $c\bar{\Delta}^{(n+1)} = 0$. Equation (1.9) determines a unique collection of maps $c\check{\psi} \in \underline{\Lambda\text{-mod}}\big(\mathfrak{a}(U, V), \widehat{T\mathfrak{b}}((U, X)\phi, (V, Y)\phi)\big)^d$. It remains to verify that it is a coderivation. We have to prove that

$$(a)(c\check{\psi})\Delta_{\mathfrak{b}} = (a)\Delta_{\mathfrak{a}}[(\_ \boxtimes X)\phi \otimes (\_)(c\check{\psi}) + (\_)(c\check{\psi}) \otimes (\_ \boxtimes Y)\phi].$$

The case $n = 0$ being obvious, assume that $n \geqslant 1$. The sum in (1.9) goes from $k = 2$ to n. Correspondingly,

$$(a)(c\check{\psi})\Delta = (a\Delta)[(\_ \boxtimes c_{(1)})\phi \otimes (\_ \boxtimes c_{(2)})\phi] - \sum_{k=2}^{n} [(a\Delta) \boxtimes (c_{\bar{1}}\check{\psi} \otimes \cdots \otimes c_{\bar{k}}\check{\psi})\Delta_{T\operatorname{Coder}}]\tau_{(23)}(\operatorname{ev} \otimes \operatorname{ev}).$$

Here according to Sweedler's notation $c_{(1)} \otimes c_{(2)} = c\Delta$. Similarly, $c_{\bar{1}} \otimes \cdots \otimes c_{\bar{k}} = c\bar{\Delta}^{(k)}$. Recall the middle four interchange $[(a \otimes b) \boxtimes (c \otimes d)]\tau_{(23)} = (-1)^{bc}(a \boxtimes c) \otimes (b \boxtimes d)$. The above expression has to be equal to

$$\begin{aligned}(a\Delta)\Big\{(\_ \boxtimes 1_X)\phi \otimes \big[(\_ \boxtimes c)\phi - \sum_{k=2}^{n} \big(\_ \boxtimes (c_{\bar{1}}\check{\psi} \otimes \cdots \otimes c_{\bar{k}}\check{\psi})\big) \operatorname{ev}\big]\Big\} \\ + (a\Delta)\Big\{\big[(\_ \boxtimes c)\phi - \sum_{k=2}^{n} \big(\_ \boxtimes (c_{\bar{1}}\check{\psi} \otimes \cdots \otimes c_{\bar{k}}\check{\psi})\big) \operatorname{ev}\big] \otimes (\_ \boxtimes 1_Y)\phi\Big\}.\end{aligned}$$

Canceling the above terms we come to identity to be checked

$$(a\Delta)[(\_\boxtimes c_{\bar{1}})\phi\otimes(\_\boxtimes c_{\bar{2}})\phi]=\sum_{k=2}^{n}\big[(a\Delta)\boxtimes(c_{\bar{1}}\check{\psi}\otimes\cdots\otimes c_{\bar{k}}\check{\psi})\bar{\Delta}_{T\,\mathrm{Coder}}\big]\tau_{(23)}(\mathrm{ev}\otimes\mathrm{ev}).\quad(1.10)$$

The right hand side equals

$$\begin{aligned}
&\sum_{k=2}^{n}\sum_{i=1}^{k-1}\Big\{(a\Delta)\boxtimes\big[(c_{\bar{1}}\check{\psi}\otimes\cdots\otimes c_{\bar{i}}\check{\psi})\bigotimes(c_{\overline{i+1}}\check{\psi}\otimes\cdots\otimes c_{\bar{k}}\check{\psi})\big]\Big\}\tau_{(23)}(\mathrm{ev}\otimes\mathrm{ev})\\
&=\sum_{k=2}^{n}\sum_{i=1}^{k-1}(a\Delta)\Big\{\big[_\boxtimes(c_{\bar{1}}\check{\psi}\otimes\cdots\otimes c_{\bar{i}}\check{\psi})\big]\mathrm{ev}\otimes\big[_\boxtimes(c_{\overline{i+1}}\check{\psi}\otimes\cdots\otimes c_{\bar{k}}\check{\psi})\big]\,\mathrm{ev}\Big\}\\
&=\sum_{i=1}^{n}\sum_{j=1}^{n}(a\Delta)\Big\{\big[_\boxtimes(c_{\bar{1}}\check{\psi}\otimes\cdots\otimes c_{\bar{i}}\check{\psi})\big]\mathrm{ev}\otimes\big[_\boxtimes(c_{\overline{i+1}}\check{\psi}\otimes\cdots\otimes c_{\overline{i+j}}\check{\psi})\big]\,\mathrm{ev}\Big\}\\
&=(a\Delta)[(_)(c_{\bar{1}}F)\otimes(_)(c_{\bar{2}}F)],
\end{aligned}$$

where

$$(a)(cF)=\sum_{i=1}^{n}\big[a\boxtimes(c_{\bar{1}}\check{\psi}\otimes\cdots\otimes c_{\bar{i}}\check{\psi})\big]\mathrm{ev}=(a)(c\check{\psi})+\sum_{i=2}^{n}\big[a\boxtimes(c_{\bar{1}}\check{\psi}\otimes\cdots\otimes c_{\bar{i}}\check{\psi})\big]\mathrm{ev}=(a\boxtimes c)\phi$$

due to (1.9). Hence the right hand side of (1.10) equals $(a\Delta)[(\_\boxtimes c_{\bar{1}})\phi\otimes(\_\boxtimes c_{\bar{2}})\phi]$, which is the left hand side of (1.10). ∎

Let $\mathfrak{a}$ be a conilpotent cocategory and let $\mathfrak{b}$, $\mathfrak{c}$ be quivers. Consider the cofunctor given by the upper right path in the diagram

$$\begin{array}{ccc}
\mathfrak{a}\boxtimes T\,\mathrm{Coder}(\mathfrak{a},T\mathfrak{b})\boxtimes T\,\mathrm{Coder}(T\mathfrak{b},T\mathfrak{c}) & \xrightarrow{\mathrm{ev}\boxtimes 1} & T\mathfrak{b}\boxtimes T\,\mathrm{Coder}(T\mathfrak{b},T\mathfrak{c})\\
\Big\downarrow{\scriptstyle 1\boxtimes M} & = & \Big\downarrow{\scriptstyle \mathrm{ev}}\\
\mathfrak{a}\boxtimes T\,\mathrm{Coder}(\mathfrak{a},T\mathfrak{c}) & \xrightarrow{\qquad\mathrm{ev}\qquad} & T\mathfrak{c}
\end{array}$$

By Proposition 1.14 there is a unique augmentation preserving cofunctor

$$M: T\,\mathrm{Coder}(\mathfrak{a},T\mathfrak{b})\boxtimes T\,\mathrm{Coder}(T\mathfrak{b},T\mathfrak{c})\to T\,\mathrm{Coder}(\mathfrak{a},T\mathfrak{c}).$$

Denote by $\mathbb{1}$ the unit object $\boxtimes^0$ of the monoidal category of nilpotent cocategories, that is, $\mathrm{Ob}\,\mathbb{1}=\{*\}$, $\mathbb{1}(*,*)=\Lambda$. Denote by $\mathbf{r}:\mathfrak{a}\boxtimes\mathbb{1}\to\mathfrak{a}$ and $\mathbf{l}:\mathbb{1}\boxtimes\mathfrak{a}\to\mathfrak{a}$ the corresponding natural isomorphisms. By Proposition 1.14 there exists a unique augmentation preserving cofunctor $\eta_{T\mathfrak{b}}:\mathbb{1}\to T\,\mathrm{Coder}(T\mathfrak{b},T\mathfrak{b})$, such that

$$\mathbf{r}=\big(T\mathfrak{b}\boxtimes\mathbb{1}\xrightarrow{1\boxtimes\eta_{T\mathfrak{b}}}T\mathfrak{b}\boxtimes T\,\mathrm{Coder}(T\mathfrak{b},T\mathfrak{b})\xrightarrow{\mathrm{ev}}T\mathfrak{b}\big).$$

Namely, the object $*\in\mathrm{Ob}\,\mathbb{1}$ goes to the identity homomorphism $\mathrm{id}_{T\mathfrak{b}}:T\mathfrak{b}\to T\mathfrak{b}$.

The following statement (published as [Lyubashenko, 2003, Proposition 4.1]) follows from Proposition 1.14.

1.15. Proposition. *The multiplication M is associative and η is its two-sided unit:*

$$\begin{array}{ccc} T\mathrm{Coder}(\mathfrak{a},T\mathfrak{b})\boxtimes T\mathrm{Coder}(T\mathfrak{b},T\mathfrak{c})\boxtimes T\mathrm{Coder}(T\mathfrak{c},T\mathfrak{d}) & \xrightarrow{M\boxtimes 1} & T\mathrm{Coder}(\mathfrak{a},T\mathfrak{c})\boxtimes T\mathrm{Coder}(T\mathfrak{c},T\mathfrak{d}) \\ \downarrow{\scriptstyle 1\boxtimes M} & & \downarrow{\scriptstyle M} \\ T\mathrm{Coder}(\mathfrak{a},T\mathfrak{b})\boxtimes T\mathrm{Coder}(T\mathfrak{b},T\mathfrak{d}) & \xrightarrow{\qquad M\qquad} & T\mathrm{Coder}(\mathfrak{a},T\mathfrak{d}) \end{array}$$

The multiplication M is computed explicitly in [Lyubashenko, 2003, §4], see, in particular, Examples 4.2 there.

2. Filtered cocategories

Let $\mathbb{L}$ be a partially ordered commutative monoid with the operation $+$ and neutral element 0. Of course, we assume that $a \leqslant b$, $c \leqslant d$ imply $a + c \leqslant b + d$. The subsets

$$\mathbb{L}_+ = \{l \in \mathbb{L} \mid l \geqslant 0\},$$
$$\mathbb{L}_- = \{l \in \mathbb{L} \mid l \leqslant 0\}$$

are submonoids. Clearly, $\mathbb{L}_+ \cap \mathbb{L}_- = \{0\}$. We require that

$$\mathbb{L}_{++} = \{l \in \mathbb{L} \mid l > 0\} = \mathbb{L}_+ - 0$$

were non-empty. We assume that $\mathbb{L}$ satisfies the following conditions:

(i) for all $a, b \in \mathbb{L}$ there is $c \in \mathbb{L}$ such that $a \leqslant c$, $b \leqslant c$ (that is, $(\mathbb{L}, \leqslant)$ is directed);

(ii) for all $a, b \in \mathbb{L}$ there is $c \in \mathbb{L}$ such that $c \leqslant a$, $c \leqslant b$ (that is, $\mathbb{L}^{\mathrm{op}}$ is directed);

(iii) for all $a, b \in \mathbb{L}$ there is $c \in \mathbb{L}$ such that $a + c \geqslant b$.

This generalizes the assumptions of [De Deken, Lowen, 2018]. If $\mathbb{L}$ is a directed group (satisfies (i)), then $\mathbb{L}$ satisfies (ii) and (iii) as well for obvious reasons.

The symmetric monoidal category of $\mathbb{Z}$-graded abelian groups (with the usual signed symmetry) is denoted grAb. An $\mathbb{L}$-filtered graded abelian group is a $\mathbb{Z}$-graded abelian group M together with, for every $l \in \mathbb{L}$, a graded subgroup $\mathcal{F}^l M$ such that $a \leqslant b \in \mathbb{L}$ implies that $\mathcal{F}^a M \supset \mathcal{F}^b M$ and $\cup_{l\in\mathbb{L}} \mathcal{F}^l M = M$. The symmetric multicategory $\widehat{\mathsf{grAb}_{\mathbb{L}}}$ of $\mathbb{L}$-filtered graded abelian groups is formed by polylinear maps of certain degree preserving the filtration:

$$\widehat{\mathsf{grAb}_{\mathbb{L}}}(M_1, \dots, M_n; N)^d = \{(\text{polylinear maps } f : M_1^{k_1} \times \dots \times M_n^{k_n} \to N^{k_1+\dots+k_n+d})_{k_i\in\mathbb{Z}} \mid$$
$$\mid (\mathcal{F}^{l_1} M_1^{k_1} \times \dots \times \mathcal{F}^{l_n} M_n^{k_n}) f \subset \mathcal{F}^{l_1+\dots+l_n} N^{k_1+\dots+k_n+d}\},$$

$n \geqslant 1$. The sign for composition is the same as in [Bespalov, Lyubashenko, Manzyuk, 2008, Example 3.17]. This multicategory is representable [Bespalov, Lyubashenko, Manzyuk, 2008, Definition 3.23] (see also [Hermida, 2000, Definition 8.3]) by a symmetric monoidal

category which we denote $\mathsf{grAb}_{\mathbb{L}}$. This follows from a similar statement for the closed multicategory $\widehat{\mathsf{Ab}}$ of abelian groups. One deduces the tensor product of a family M_1, …, M_n, $n \geqslant 1$, as the tensor product of $\mathbb{Z}$-graded abelian groups M_i, equipped with the filtration [De Deken, Lowen, 2018, (2)]

$$\mathcal{F}^l(\otimes_{i=1}^n M_i) = \mathrm{Im}(\oplus_{l_1+\cdots+l_n=l} \otimes_{i=1}^n \mathcal{F}^{l_i} M_i \to \otimes_{i=1}^n M_i). \tag{2.1}$$

Thus, $\widehat{\mathsf{grAb}_{\mathbb{L}}}(M_1, \ldots, M_n; N)$ is naturally isomorphic to $\mathsf{grAb}_{\mathbb{L}}(M_1 \otimes \cdots \otimes M_n, N)$ for $n \geqslant 1$ (more in [Bespalov, Lyubashenko, Manzyuk, 2008, Theorem 3.24]). The unit object is $\mathbb{Z}$, concentrated in degree 0, equipped with the filtration

$$\mathcal{F}^l\mathbb{Z} = \begin{cases} \mathbb{Z}, & \text{if } l \leqslant 0, \\ 0, & \text{otherwise.} \end{cases}$$

We define $\widehat{\mathsf{grAb}_{\mathbb{L}}}(; N)$ as $\mathsf{grAb}_{\mathbb{L}}(\mathbb{Z}, N)$ in order to keep representability.

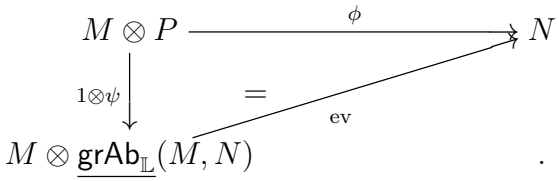
The monoidal category $\mathsf{grAb}_{\mathbb{L}}$ is symmetric with the signed symmetry of $\mathbb{Z}$-graded abelian groups. Furthermore, it is closed. In fact, let $M, N \in \mathsf{grAb}_{\mathbb{L}}$. Associate with them a new graded $\mathbb{L}$-filtered abelian group $\underline{\mathsf{grAb}_{\mathbb{L}}}(M, N)$ with

$$\mathcal{F}^l\underline{\mathsf{grAb}_{\mathbb{L}}}(M, N)^d = \{f \in \underline{\mathsf{grAb}}(M, N)^d \mid \forall \lambda \in \mathbb{L}\ \forall k \in \mathbb{Z}\ (\mathcal{F}^\lambda M^k)f \subset \mathcal{F}^{\lambda+l} N^{k+d}\},$$

the inner hom. The evaluation

$$\mathrm{ev} : M \otimes \underline{\mathsf{grAb}_{\mathbb{L}}}(M, N) \to N, \quad m \otimes f \mapsto (m)f,$$

is a morphism of $\mathsf{grAb}_{\mathbb{L}}$, and it turns $\mathsf{grAb}_{\mathbb{L}}$ into a closed symmetric monoidal category. Indeed, let $\phi : M \otimes P \to N \in \mathsf{grAb}_{\mathbb{L}}$. To any $p \in P^d$, $d \in \mathbb{Z}$, assign a degree d map $\psi(p) : M \to N$, $m \mapsto \phi(m \otimes p)$. If $p \in \mathcal{F}^l P^d$, then $\psi(p) \in \mathcal{F}^l\underline{\mathsf{grAb}_{\mathbb{L}}}(M, N)^d$. Hence a map $\psi : P \to \underline{\mathsf{grAb}_{\mathbb{L}}}(M, N) \in \mathsf{grAb}_{\mathbb{L}}$ such that



Vice versa, given ψ one obtains ϕ as the composition $(1 \otimes \psi) \cdot \mathrm{ev}$. The two maps $\phi \longleftrightarrow \psi$ are inverse to each other, and $\mathsf{grAb}_{\mathbb{L}}$ is closed.

According to [Bespalov, Lyubashenko, Manzyuk, 2008, Proposition 4.8] the symmetric multicategory $\widehat{\mathsf{grAb}_{\mathbb{L}}}$ is closed as well. It is easy to describe the inner hom $\underline{\widehat{\mathsf{grAb}_{\mathbb{L}}}}(M_1, \ldots, M_n; N) \in \mathrm{Ob}\,\widehat{\mathsf{grAb}_{\mathbb{L}}}$ via

$$\mathcal{F}^l\underline{\widehat{\mathsf{grAb}_{\mathbb{L}}}}(M_1, \ldots, M_n; N)^d = \{(\text{polylinear maps } f : M_1^{k_1} \times \cdots \times M_n^{k_n} \to N^{k_1+\cdots+k_n+d})_{k_i \in \mathbb{Z}} \mid$$
$$\mid (\mathcal{F}^{l_1} M_1^{k_1} \times \cdots \times \mathcal{F}^{l_n} M_n^{k_n})f \subset \mathcal{F}^{l_1+\cdots+l_n+l} N^{k_1+\cdots+k_n+d}\}.$$

The corresponding evaluation is

$$\mathrm{ev} : M_1, \dots, M_n, \widehat{\underline{\mathsf{grAb}_{\mathbb{L}}}}(M_1, \dots, M_n; N) \to N, \quad (m_1, \dots, m_n, f) \mapsto (m_1, \dots, m_n)f.$$

A commutative $\mathbb{L}$-filtered graded ring Λ is a commutative monoid (commutative algebra) in $\mathsf{grAb}_{\mathbb{L}}$. Modules over Λ in $\mathsf{grAb}_{\mathbb{L}}$ are called $\mathbb{L}$-filtered $\mathbb{Z}$-graded Λ-modules and are identified with commutative Λ-bimodules (for short, Λ-modules). In examples of interest (see Example 2.10) Λ is $2\mathbb{Z}$-graded, so the commutativity issues for it are the same as in the non-graded case.

Due to condition (i) a filtration $(\mathcal{F}^\lambda M^k)_{\lambda\in\mathbb{L}}$ on the graded k-th component M^k viewed as a basis of neighborhoods of the origin defines a uniform structure on M^k with the entourages $\{(x, y) \in M \times M \mid x - y \in \mathcal{F}^\lambda M^k\}$. Standard properties of uniform structures are listed in [Bourbaki, 1971, Chap. II, §1, §2].

2.1. Proposition. *With the above uniform structure*

(a) An element of $\underline{\mathsf{grAb}_{\mathbb{L}}}(M, N)^d$ *is a family of uniformly continuous maps* $M^k \to N^{k+d}$.

(b) Each $f \in \underline{\mathsf{grAb}_{\mathbb{L}}}(M_1, \dots, M_n; N)^d$ *is a family of continuous maps* $f : M_1^{k_1} \times \cdots \times M_n^{k_n} \to N^{k_1+\cdots+k_n+d}$, *where* $M_i^{k_i}$, N^k *are given the topology, associated with the uniform structure [Bourbaki, 1971, Chap. II, §1, n. 2, Definition 3].*

(c) If $\mathbb{L} = \mathbb{L}_+$, *then each* $f \in \underline{\mathsf{grAb}_{\mathbb{L}}}(M_1, \dots, M_n; N)^d$ *is a family of uniformly continuous maps* $f : M_1^{k_1} \times \cdots \times M_n^{k_n} \to N^{k_1+\cdots+k_n+d}$.

Proof. (a) Let $f \in \mathcal{F}^l\underline{\mathsf{grAb}_{\mathbb{L}}}(M, N)^d$. For any $h \in \mathbb{L}$ there exists $\lambda \in \mathbb{L}$ such that $l+\lambda \geqslant h$ by condition (iii). Then for arbitrary points $x, y \in M^k$ such that $x - y \in \mathcal{F}^\lambda M^k$ we have $f(x) - f(y) = f(x - y) \in \mathcal{F}^{l+\lambda}N^{k+d} \subset \mathcal{F}^h N^{k+d}$.

(b) Fix a point $(y_1, \dots, y_n) \in M_1^{k_1} \times \cdots \times M_n^{k_n}$. There are $c_i \in \mathbb{L}$ such that $y_i \in \mathcal{F}^{c_i} M_i^{k_i}$. For an arbitrary $\lambda \in \mathbb{L}$ take $\lambda_i \in \mathbb{L}$ such that

$$\lambda_i \geqslant c_i, \qquad \lambda_i + \sum_{j\neq i} c_j \geqslant \lambda.$$

Consider the neighborhood of y_i

$$\{x_i \in M_i^{k_i} \mid x_i - y_i \in \mathcal{F}^{\lambda_i} M_i^{k_i}\} \subset \mathcal{F}^{c_i} M_i^{k_i}.$$

For x_i from this neighborhood the element $f(x_1, \dots, x_n)$ is in neighborhood of $f(y_1, \dots, y_n)$, namely,

$$\begin{aligned} f(x_1, \dots, x_n) - f(y_1, \dots, y_n) =& f(x_1 - y_1, x_2, \dots, x_n) + f(y_1, x_2 - y_2, x_3, \dots, x_n) \\ &+ \cdots + f(y_1, \dots, y_{n-1}, x_n - y_n) \in \\ \in \mathcal{F}^{\lambda_1+c_2+\cdots+c_n} N^k &+ \mathcal{F}^{c_1+\lambda_2+c_2+\cdots+c_n} N^k + \cdots + \mathcal{F}^{c_1+\cdots+c_{n-1}+\lambda_n} N^k \subset \mathcal{F}^\lambda N^k, \quad (2.2) \end{aligned}$$

where $k = k_1 + \cdots + k_n + d$.

(c) For any $\lambda \in \mathbb{L}$ and any points $(x_1, \dots, x_n), (y_1, \dots, y_n) \in M_1^{k_1} \times \cdots \times M_n^{k_n}$ if $x_i - y_i \in \mathcal{F}^\lambda M_i^{k_i}$, $1 \leqslant i \leqslant n$, then $f(x_1, \dots, x_n) - f(y_1, \dots, y_n) \in \mathcal{F}^\lambda N^{k_1+\cdots+k_n+d}$ similarly to (2.2). ∎

2.2. Completion of a filtered graded abelian group. The notion of complete filtered abelian group is a particular case of a complete uniform space [Bourbaki, 1971, Chap. II, §3, n.3, Def. 3]. There is the notion of separated completion (from now on *completion*) $\hat{M} = (\hat{M}^k)$ of a uniform space $M = (M^k)$ [Bourbaki, 1971, Chap. II, §3, n.7, Th. 3]. It consists of minimal Cauchy filters on $M = (M^k)$ [Bourbaki, 1971, Chap. II, §3, n.2]. It is known that any Cauchy filter F contains a unique minimal Cauchy filter $\underline{F}$ [Bourbaki, 1971, Chap. II, §3, n.2, Prop. 5]. A base of the filter $\underline{F}$ can be obtained as a family $\{A + \mathcal{F}^\lambda M \mid A \in B,\ \lambda \in \mathbb{L}\}$, where B is a base of the filter F.

Consider now the graded abelian group $\widehat{M} = \lim_{\lambda\in\mathbb{L}^{\mathrm{op}}}(M/\mathcal{F}^\lambda M)$ equipped with filtration

$$\mathcal{F}^l\widehat{M} = \lim_{\lambda\in\mathbb{L}^{\mathrm{op}}}((\mathcal{F}^l M + \mathcal{F}^\lambda M)/\mathcal{F}^\lambda M). \tag{2.3}$$

We understand the first limit as terminal cone on the functor $\mathbb{L}^{\mathrm{op}} \to \mathsf{grAb}$, $\lambda \mapsto M/\mathcal{F}^\lambda M$. In our assumptions the non-empty subsemigroup $\mathbb{L}_{++}$ is a final subset of poset $\mathbb{L}$. Hence, $\lim_{\lambda\in\mathbb{L}^{\mathrm{op}}}(M/\mathcal{F}^\lambda M) = \lim_{\lambda\in\mathbb{L}^{\mathrm{op}}_{++}}(M/\mathcal{F}^\lambda M)$. We are going to prove that $\widehat{M}$ coincides with $\hat{M}$. Until done we distinguish the two notations.

2.3. Proposition. *When $\mathbb{L}$ satisfies condition (i), the filtered graded abelian group $\widehat{M}$ is complete.*

Proof (Seems known). It suffices to look at a graded component of M which we still denote M. The definition of completeness can be given also via Cauchy nets, namely, we have to prove that any Cauchy net in $\widehat{M}$ converges. A net is a mapping $x : D \to \widehat{M} = \lim_{\lambda\in\mathbb{L}^{\mathrm{op}}}(M/\mathcal{F}^\lambda M)$, $d \mapsto x^d = ([x^d_\lambda])_{\lambda\in\mathbb{L}}$, where D is a preordered directed set. Classes $[x^d_\lambda] \in M/\mathcal{F}^\lambda M$ lift to elements $x^d_\lambda \in M$ such that for any $d \in D$ and for all $a \leqslant b \in \mathbb{L}$ we have $x^d_a - x^d_b \in \mathcal{F}^a M$. The net x is Cauchy iff for every $l \in \mathbb{L}$ there is $N = N(l) \in D$ such that for all $n, m \geqslant N \in D$ we have $x^n - x^m \in \mathcal{F}^l\widehat{M}$. The last condition reads: for every $\lambda \in \mathbb{L}$ we have $x^n_\lambda - x^m_\lambda \in \mathcal{F}^l M + \mathcal{F}^\lambda M$.

Let x as above be a Cauchy net. Consider the collection $y = (y_\lambda) \overset{\mathrm{def}}{=} (x^{N(\lambda)}_\lambda)_{\lambda\in\mathbb{L}}$. Let us show that $y \in \widehat{M}$. Recall that for $a \leqslant b \in \mathbb{L}$ there is $N \in D$ such that $N \geqslant N(a)$, $N \geqslant N(b)$. Then $y_a \equiv x^N_a \mod \mathcal{F}^a M$, $y_b \equiv x^N_b \mod \mathcal{F}^b M$, $x^N_a \equiv x^N_b \mod \mathcal{F}^a M$. Hence $y_a \equiv y_b \mod \mathcal{F}^a M$ and $y \in \widehat{M}$. It follows from the condition in the first paragraph that x converges to y, thus $\widehat{M}$ is complete. ∎

2.4. Proposition. *When $\mathbb{L}$ satisfies condition (i), the separated completion $\hat{M}$ of a filtered graded abelian group M coincides with $\widehat{M}$. The filtration*

$$\mathcal{F}^l\hat{M} = \{F \in \hat{M} \mid \exists\, 0 \in A \in F \quad A \subset \mathcal{F}^l M\}, \quad l \in \mathbb{L},$$

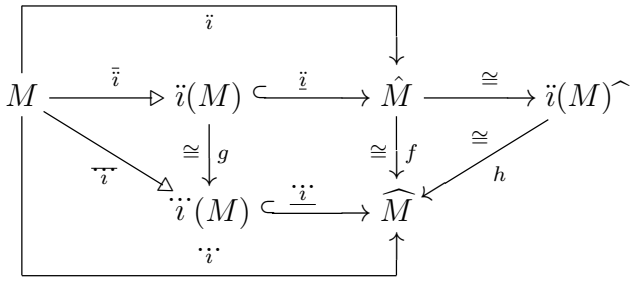
on $\hat{M}$ identifies with filtration (2.3) on $\widehat{M}$.

Proof. The canonical mapping $\ddot{\imath} : M \to \widehat{M}$ is uniformly continuous since for $z \in \mathcal{F}^l M$ we have $\ddot{\imath}(z) \in \mathcal{F}^l\widehat{M}$. Moreover, the filtration on M is a preimage of the filtration on $\widehat{M}$ hence preimage of the filtration on its subset $\ddot{\imath}(M)$. In fact, the image $\ddot{\imath}(z)$ of $z \in M$ is

in $\mathcal{F}^l\widehat{M}$ iff for all $\lambda \in \mathbb{L}$ we have $z \in \mathcal{F}^l M + \mathcal{F}^\lambda M$. For $\lambda = l$ we get $z \in \mathcal{F}^l M$. Furthermore, $\widehat{M}$ is separated. In fact, if $x = ([x_\lambda])_{\lambda\in\mathbb{L}}$ belongs to all $\mathcal{F}^l\widehat{M}$ then $x_\lambda \in \mathcal{F}^l M + \mathcal{F}^\lambda M$ for all $l, \lambda \in \mathbb{L}$, hence, $x_\lambda \in \mathcal{F}^\lambda M$ for all $\lambda \in \mathbb{L}$, that is, $x = 0$.

Let us prove that the image $\ddot{i}(M)$ is everywhere dense in $\widehat{M}$. For any $x = ([x_\lambda]) \in \widehat{M}$ and any $l \in \mathbb{L}$ we have to provide an element $y \in M$ such that $x - \ddot{i}(y) \in \mathcal{F}^l\widehat{M}$. The last condition reads: for all $\lambda \in \mathbb{L}$ we have $x_\lambda - y \in \mathcal{F}^l M + \mathcal{F}^\lambda M$. Take $y = x_l$. By assumption for all $l, \lambda \in \mathbb{L}$ there is $c \in \mathbb{L}$ such that $l \leqslant c$, $\lambda \leqslant c$. Hence, $x_\lambda - y = x_\lambda - x_c + x_c - x_l \in \mathcal{F}^\lambda M + \mathcal{F}^l M$ as required.

Now we can construct the following commutative diagram



where $\ddot{i} : M \to \hat{M}$ is the canonical mapping, sending a point to the filter of neighborhoods of this point. It is denoted by i in [Bourbaki, 1971, Chap. II, §3, n.7, 2)]. The isomorphism $\hat{M} \to \ddot{i}(M)\hat{}$ follows by Propositions 12.1° and 13 of [Bourbaki, 1971, Chap. II, §3, n.7]. The morphism g exists by Proposition 16 and it is invertible by Proposition 17 of [Bourbaki, 1971, Chap. II, §3, n.8]. The isomorphism h exists by [Bourbaki, 1971, Chap. II, §3, n.7, Prop. 13]. Hence an isomorphism f.

The filter $\mathcal{O} = \ddot{i}(0) \in \hat{M}$ of neighborhoods of $0 \in M$ is a minimal Cauchy filter. It has the base $\{\mathcal{F}^\lambda M \mid \lambda \in \mathbb{L}\}$. Filtration on $\hat{M}$ consists of

$$\begin{aligned}\mathcal{F}^l\hat{M} &= \{F \in \hat{M} \mid \exists\, A \in F \cap \mathcal{O} \;\; A - A \subset \mathcal{F}^l M\}\\ &= \{F \in \hat{M} \mid \exists\, A \in F\; \exists \lambda \in \mathbb{L} \;\; A \supset \mathcal{F}^\lambda M,\; A - A \subset \mathcal{F}^l M\}\\ &= \{F \in \hat{M} \mid \exists\, 0 \in A \in F \;\; A \subset \mathcal{F}^l M\}\end{aligned}$$

[Bourbaki, 1971, Chap. II, §3, n.7, 1)] (since Cauchy filter F is minimal, its base consists of sets invariant under addition of $\mathcal{F}^\lambda M$ for some $\lambda \in \mathbb{L}$, which we may assume $\geqslant l$).

Both $\mathcal{F}^l\hat{M}$ and $\mathcal{F}^l\widehat{M}$ induce on M the same subspace $\mathcal{F}^l M$ via pull-back with bases $\ddot{i} : M \to \hat{M}$ and $\ddot{\ddot{i}} : M \to \widehat{M}$ [Bourbaki, 1971, Chap. II, §3, n.7, 2)]. The images $\ddot{i}(M) \subset \hat{M}$, $\ddot{\ddot{i}}(M) \subset \widehat{M}$ are dense [Bourbaki, 1971, Chap. II, §3, n.7, 3)], hence, filtrations $\mathcal{F}^l\hat{M}$ and $\mathcal{F}^l\widehat{M}$ are taken to each other under the isomorphism $f : \hat{M} \to \widehat{M}$ and its inverse.

■

From now on we do not distinguish $\hat{M}$ and $\widehat{M}$.

The mapping $M \mapsto \hat{M}$ extends to the completion functor $\hat{\cdot} : \mathsf{grAb}_{\mathbb{L}} \to \mathsf{grAb}_{\mathbb{L}}$ [Bourbaki, 1971, Chap. II, §3, n.7, Prop. 15] in a unique way so that the maps $\ddot{i}_M : M \to \hat{M}$ form

a natural transformation. A filtered graded abelian group M is complete, when the canonical map $\ddot{\imath} : M \to \hat{M}$ is an isomorphism. The $\mathsf{grAb}_{\mathbb{L}}$-category $\mathsf{cgrAb}_{\mathbb{L}}$ of complete $\mathbb{L}$-filtered graded abelian groups is a reflective subcategory of $\mathsf{grAb}_{\mathbb{L}}$. This follows by the remark that complete topological abelian groups form a reflective subcategory of the category of topological abelian groups. Thus by [Borceux, 1994, Corollary 4.2.4] (see enriched version at the end of Chapter 1 of [Kelly, 1982]) the completion functor is an idempotent monad. In particular, for the unit of this monad $\ddot{\imath}_M : M \to \hat{M}$, the morphisms $\ddot{\imath}_{\hat{M}} = \widehat{\ddot{\imath}_M} : \hat{M} \to \hat{\hat{M}}$ are inverse to the multiplication $\mu_M : \hat{\hat{M}} \to \hat{M}$ (cf. [De Deken, Lowen, 2018, Lemma 2.24]). It follows from Appendix A that the reflective subcategory $\mathsf{cgrAb}_{\mathbb{L}}$ is symmetric monoidal with the monoidal product $M \hat{\otimes} N \stackrel{\text{def}}{=} \widehat{M \otimes N}$. The unit object is still $\mathbb{Z}$. We extend the functor $\hat{\otimes}^n : \mathsf{cgrAb}_{\mathbb{L}}^n \to \mathsf{cgrAb}_{\mathbb{L}}$ to $\hat{\otimes}^n : \mathsf{grAb}_{\mathbb{L}}^n \to \mathsf{cgrAb}_{\mathbb{L}}$ via the same recipe $\hat{\otimes}_{i=1}^n M_i \stackrel{\text{def}}{=} \widehat{\otimes_{i=1}^n M_i}$.

2.5. Proposition. *For $n \geqslant 1$ there is a natural transformation $\phi^n : \otimes_{i=1}^n \widehat{M_i} \to \widehat{\otimes_{i=1}^n M_i} : \mathsf{grAb}_{\mathbb{L}}^n \to \mathsf{grAb}_{\mathbb{L}}$.*

Proof. Let M_i, $1 \leqslant i \leqslant n$, be filtered abelian groups. Let $F_i \in \hat{M}_i$ be minimal Cauchy filters in M_i, $1 \leqslant i \leqslant n$. Denote $M = \otimes_{i=1}^n M_i$. Define a basis B of a filter F in M as

$$B = \{A_1 \otimes A_2 \otimes \cdots \otimes A_n \mid \forall i\, A_i \in F_i\},$$

where $A_1 \otimes A_2 \otimes \cdots \otimes A_n = \{x_1 \otimes x_2 \otimes \cdots \otimes x_n \mid \forall i\, x_i \in A_i\}$. Let us prove that F is a Cauchy filter. Given $\lambda \in \mathbb{L}$, take for $1 \leqslant i \leqslant n$ arbitrary elements $a_i \in \mathbb{L}$, take $A_i \in F_i$ such that $A_i - A_i \equiv \{x - y \mid x, y \in A_i\} \subset \mathcal{F}^{a_i} M_i$, take arbitrary elements $y_i \in A_i$. Let $b_i \in \mathbb{L}$ be such that $y_i \in \mathcal{F}^{b_i} M_i$. Let $c_i \in \mathbb{L}$ be such that $c_i \leqslant a_i$ and $c_i \leqslant b_i$. Then $A_i \subset \mathcal{F}^{c_i}\mathbb{L}$ since any $x \in A_i$ can be presented as $x = x - y_i + y_i \in \mathcal{F}^{c_i} M_i + \mathcal{F}^{b_i} M_i \subset \mathcal{F}^{c_i} M_i$. Let $\lambda_i \in \mathbb{L}$, $1 \leqslant i \leqslant n$, be such that

$$\lambda_i + \sum_{j \neq i} c_j \geqslant \lambda.$$

Let $B_i \in F_i$ satisfy $B_i - B_i \subset \mathcal{F}^{\lambda_i} M_i$. Then the set $S = (A_1 \cap B_1) \otimes \cdots \otimes (A_n \cap B_n) \in F$ satisfies $S - S \subset \mathcal{F}^{\lambda} M$. In fact, for $x_i, y_i \in A_i \cap B_i$ we have

$$\begin{aligned} x_1 \otimes \cdots \otimes x_n - y_1 \otimes \cdots \otimes y_n &= (x_1 - y_1) \otimes x_2 \otimes \cdots \otimes x_n + y_1 \otimes (x_2 - y_2) \otimes x_3 \otimes \cdots \otimes x_n \\ &\quad + \cdots + y_1 \otimes \cdots \otimes y_{n-1} \otimes (x_n - y_n) \in \\ \in \mathcal{F}^{\lambda_1} M_1 \otimes \mathcal{F}^{c_2} M_2 \otimes \cdots \otimes \mathcal{F}^{c_n} M_n &+ \mathcal{F}^{c_1} M_1 \otimes \mathcal{F}^{\lambda_2} M_2 \otimes \mathcal{F}^{c_3} M_3 \otimes \cdots \otimes \mathcal{F}^{c_n} M_n + \ldots \\ &\quad + \mathcal{F}^{c_1} M_1 \otimes \cdots \otimes \mathcal{F}^{c_{n-1}} M_{n-1} \otimes \mathcal{F}^{\lambda_n} M_n \subset \mathcal{F}^{\lambda} M. \end{aligned} \tag{2.4}$$

The Cauchy filter F contains a unique minimal Cauchy filter $\underline{F}$ [Bourbaki, 1971, Chap. II, §3, n.2, Prop. 5] and we define ϕ^n as a map sending $F_1 \otimes \cdots \otimes F_n$ to $\underline{F}$. The outcome does not depend on the choices made during the construction. In fact, axioms on $\mathbb{L}$ and on filters ensure that two different choices F' and F'' for F are contained in a third choice F''' of Cauchy filter F, hence, for minimal Cauchy filters we have $\underline{F'} = \underline{F'''} = \underline{F''}$.

This is a well-defined mapping $\tilde{\phi}^n$ from the free graded abelian group generated by n-tuples $(F_1, \dots, F_n)$ to $\hat{M}$. One has

$$\tilde{\phi}^n(F_1, \dots, F_i' + F_i'', \dots, F_n) = \tilde{\phi}^n(F_1, \dots, F_i', \dots, F_n) + \tilde{\phi}^n(F_1, \dots, F_i'', \dots, F_n)$$

if $F_i', F_i'' \in \hat{M}_i^k$, $1 \leqslant i \leqslant n$, $k \in \mathbb{Z}$. In fact, sum of Cauchy filters G', G'' is defined as the filter G, generated by $A' + A''$, $A' \in G'$, $A'' \in G''$. Clearly, G is a Cauchy filter. Thus, the map $\tilde{\phi}^n$ factors through a well-defined map $\phi^n : \otimes_{i=1}^n \widehat{M_i} \to \widehat{\otimes_{i=1}^n M_i}$.

Notice that

$$\phi^n(\otimes_{i=1}^n \mathcal{F}^{\lambda_i} \hat{M}_i) \subset \mathcal{F}^{\lambda_1 + \dots + \lambda_n} \widehat{\otimes_{i=1}^n M_i}.$$

In fact, let $F_i \in \mathcal{F}^{\lambda_i} \hat{M}_i$, $1 \leqslant i \leqslant n$. Thus, F_i is a minimal Cauchy filter such that there is $0 \in A_i \in F_i$, $A_i \subset \mathcal{F}^{\lambda_i} M_i$. Then

$$\phi^n(F_1 \otimes \dots \otimes F_n) \ni A_1 \otimes \dots \otimes A_n + \mathcal{F}^{\nu}(M_1 \otimes \dots \otimes M_n)$$

for any $\nu \in \mathbb{L}$. However, $0 \in A_1 \otimes \dots \otimes A_n$ and

$$A_1 \otimes \dots \otimes A_n \subset \mathcal{F}^{\lambda_1} M_1 \otimes \dots \otimes \mathcal{F}^{\lambda_n} M_n \subset \mathcal{F}^{\lambda_1 + \dots + \lambda_n}(\otimes_{i=1}^n M_i).$$

Hence, for $\nu \geqslant \lambda_1 + \dots + \lambda_n$

$$0 \in A_1 \otimes \dots \otimes A_n + \mathcal{F}^{\nu}(M_1 \otimes \dots \otimes M_n) \subset \mathcal{F}^{\lambda_1 + \dots + \lambda_n}(\otimes_{i=1}^n M_i).$$

Therefore, $\phi^n(F_1 \otimes \dots \otimes F_n) \in \mathcal{F}^{\lambda_1 + \dots + \lambda_n} \widehat{\otimes_{i=1}^n M_i}$.

Reasonings, similar to independence of choices show that ϕ^n form a natural transformation. ∎

For $n = 0$ the version of ϕ^n is the isomorphism $\phi^0 = \ddot{\imath} : \mathbb{Z} \to \widehat{\mathbb{Z}}$. The filtered abelian group $\mathbb{Z}$ is complete due to non-emptiness of $\mathbb{L}_{++}$. In fact, $\widehat{\mathbb{Z}} = \lim_{\lambda \in \mathbb{L}_{++}^{\mathrm{op}}} \mathbb{Z} = \mathbb{Z}$.

2.6. Proposition. *The pair* $(\hat{\cdot}, \phi^\bullet) : \mathsf{grAb}_{\mathbb{L}} \to \mathsf{grAb}_{\mathbb{L}}$ *is a lax symmetric monoidal functor.*

Proof. Naturality (in the ordinary everyday usage sense) of the construction of ϕ^n leads to required condition from [Day, Street, 2003], see also diagram (2.17.2) of [Bespalov, Lyubashenko, Manzyuk, 2008]. ∎

According to [Bespalov, Lyubashenko, Manzyuk, 2008, Proposition 3.28] $\phi^\bullet$ make completion $\hat{\cdot}$ also into a symmetric multifunctor $\widehat{\mathsf{grAb}_{\mathbb{L}}} \to \widehat{\mathsf{grAb}_{\mathbb{L}}}$.

2.7. Proposition. *The canonical mapping* $\ddot{\imath} : M \to \hat{M}$ *satisfies for* $n \geqslant 0$ *the equation*

$$\left(M_1 \otimes \dots \otimes M_n \xrightarrow{\ddot{\imath}_1 \otimes \dots \otimes \ddot{\imath}_n} \hat{M}_1 \otimes \dots \otimes \hat{M}_n \xrightarrow{\phi^n} \widehat{M_1 \otimes \dots \otimes M_n}\right) = \ddot{\imath}_{M_1 \otimes \dots \otimes M_n}. \tag{2.5}$$

PROOF. For $n \geqslant 1$ we may consider a graded component of M_i, a filtered abelian group, which we denote again by M_i. Take elements $y_i \in M_i$. For some $c_i \in \mathbb{L}$ we have $y_i \in \mathcal{F}^{c_i} M_i$. The filter $\ddot{\imath}(y_i) \in \hat{M}_i$ has the base formed by $y_i + \mathcal{F}^{\lambda_i} M_i$, $\lambda_i \in \mathbb{L}$. Thus, the Cauchy filter F with the basis $(y_1 + \mathcal{F}^{\lambda_1} M_1) \otimes \cdots \otimes (y_n + \mathcal{F}^{\lambda_n} M_n)$ contains the minimal Cauchy filter $\phi^n(\ddot{\imath}(y_1) \otimes \cdots \otimes \ddot{\imath}(y_n)) \in \widehat{M_1 \otimes \cdots \otimes M_n}$. For any $\lambda \in \mathbb{L}$ there are $\lambda_i \in \mathbb{L}$ such that

$$\lambda_i \geqslant c_i, \qquad \lambda_i + \sum_{j \neq i} c_j \geqslant \lambda.$$

Using (2.4) we see that the last set is contained in $y_1 \otimes \cdots \otimes y_n + \mathcal{F}^{\lambda}(M_1 \otimes \cdots \otimes M_n)$. Hence, F contains the minimal Cauchy filter of neighborhoods of $y_1 \otimes \cdots \otimes y_n$. Therefore, $\phi^n(\ddot{\imath}(y_1) \otimes \cdots \otimes \ddot{\imath}(y_n)) = \ddot{\imath}(y_1 \otimes \cdots \otimes y_n)$ by uniqueness of the minimal Cauchy subfilter.

The case of $n = 0$ is straightforward. ∎

2.8. COMPLETE Λ-MODULES. From now on, *the graded commutative filtered ring* Λ will be **complete**.

For the moment $\mathcal{V} = \Lambda\text{-mod}_{\mathbb{L}}$ means the category of $\mathbb{L}$-filtered graded Λ-modules for a graded commutative $\mathbb{L}$-filtered ring Λ. Morphisms are grading and filtration preserving Λ-module maps. It is symmetric monoidal with the tensor product $M \otimes_\Lambda N$ equipped with filtration (2.1), where, of course, $\otimes$ has to be interpreted as $\otimes_\Lambda$, not as $\otimes_{\mathbb{Z}}$. The unit object $\mathbb{1}$ is Λ with its filtration. The category $\Lambda\text{-mod}_{\mathbb{L}}$ is closed. In fact, let $M, N \in \Lambda\text{-mod}_{\mathbb{L}}$. Associate with them a new graded $\mathbb{L}$-filtered Λ-module $\underline{\Lambda\text{-mod}}_{\mathbb{L}}(M, N)$ with

$$\mathcal{F}^l \underline{\Lambda\text{-mod}}_{\mathbb{L}}(M, N)^d = \{f \in \underline{\Lambda\text{-mod}}(M, N)^d \mid \forall \lambda \in \mathbb{L}\ \forall k \in \mathbb{Z}\ (\mathcal{F}^\lambda M^k) f \subset \mathcal{F}^{\lambda + l} N^{k+d}\},$$

the inner hom. The evaluation

$$\mathrm{ev} : M \otimes_\Lambda \underline{\Lambda\text{-mod}}_{\mathbb{L}}(M, N) \to N, \quad m \otimes f \mapsto (m)f,$$

is a morphism of $\Lambda\text{-mod}_{\mathbb{L}}$, and it turns this category into a closed symmetric monoidal one. Proof is the same as in $\mathsf{grAb}_{\mathbb{L}}$ case. All definitions and notions of Section 1 apply for this $\mathcal{V}$. Note that the uniform space associated with the product $\prod_{i \in I} M_i$ in $\Lambda\text{-mod}_{\mathbb{L}}$ (over an infinite set I) differs from product of uniform spaces M_i.

A Λ-module M is complete, when the canonical map $\ddot{\imath} : M \to \hat{M}$ is an isomorphism. The category of complete Λ-modules $\Lambda\text{-mod}^{\mathrm{c}}_{\mathbb{L}}$ is a full $\Lambda\text{-mod}_{\mathbb{L}}$-subcategory of $\Lambda\text{-mod}_{\mathbb{L}}$.

2.9. REMARK. Let $(\mathfrak{c}, \Delta, \varepsilon)$ be a cocategory. Then the completion $\hat{\mathfrak{c}}$ equipped with the comultiplication $\big(\hat{\mathfrak{c}} \xrightarrow{\widehat{\Delta}} \widehat{\mathfrak{c} \otimes \mathfrak{c}} \xrightarrow{\widehat{i \otimes i}} (\hat{\mathfrak{c}} \otimes \hat{\mathfrak{c}})\widehat{\ } \equiv \hat{\mathfrak{c}} \hat{\otimes} \hat{\mathfrak{c}}\big)$ and the counit $\hat{\mathfrak{c}} \xrightarrow{\widehat{\varepsilon}} \widehat{\mathbb{1}\mathfrak{c}} = \hat{\mathbb{1}} \hat{\mathfrak{c}}$ is a cocategory over Λ with respect to monoidal structure $\hat{\otimes}$, see Appendix A.5.1.

2.10. EXAMPLE. The universal Novikov ring

$$\Lambda_{nov}(R) = \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} e^{n_i} \mid \forall i\ a_i \in R,\ \lambda_i \in \mathbb{R},\ n_i \in \mathbb{Z},\ \lim_{i \to \infty} \lambda_i = \infty \right\}$$

contains a subring, the Novikov ring,

$$\Lambda_{0,nov}(R) = \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} e^{n_i} \in \Lambda_{nov}(R) \mid \forall i\ \lambda_i \geqslant 0 \right\}$$

The grading is determined by $\deg R = 0$, $\deg T = 0$, $\deg e = 2$ [Fukaya, Oh, Ohta, Ono, 2009, §1.7 (Conv. 4)]. $\Lambda_{nov}(R)$ is $\mathbb{R}$-filtered according to [Fukaya, Oh, Ohta, Ono, 2009, §1.7 (Conv. 6)]. The filtration is

$$\mathcal{F}^\lambda \Lambda_{nov}(R) = T^\lambda \Lambda_{0,nov}(R), \qquad \lambda \in \mathbb{R}.$$

Similarly, $\Lambda_{0,nov}(R)$ is $\mathbb{R}_{\geqslant 0}$-filtered by

$$\mathcal{F}^\lambda \Lambda_{0,nov}(R) = T^\lambda \Lambda_{0,nov}(R), \qquad \lambda \in \mathbb{R}_{\geqslant 0}.$$

These rings are complete [*ibid*].

2.11. Complete cocategories. The set-up of Appendix A applies well to symmetric monoidal $\Lambda\text{-mod}_{\mathbb{L}}$-category $\mathcal{D} = (\Lambda\text{-mod}_{\mathbb{L}}\text{-}\mathsf{Quiv}, \boxtimes^I)$ and its reflective subcategory of complete quivers $\mathcal{C} = \Lambda\text{-mod}^{\mathrm{c}}_{\mathbb{L}}\text{-}\mathsf{Quiv}$. According to Proposition A.1 the category $\mathcal{C}$ is lax symmetric monoidal with the product $\widehat{\boxtimes}^{i\in I}\mathfrak{a}_i = \widehat{\boxtimes^{i\in I}\mathfrak{a}_i}$. The completion of a filtered quiver $\mathfrak{a} \in \operatorname{Ob}\mathcal{D}$ is given by the quiver $\hat{\mathfrak{a}} \in \operatorname{Ob}\mathcal{C}$ with $\operatorname{Ob}\hat{\mathfrak{a}} = \operatorname{Ob}\mathfrak{a}$ and $\hat{\mathfrak{a}}(X,Y) = \widehat{\mathfrak{a}(X,Y)}$ for $X, Y \in \operatorname{Ob}\mathfrak{a}$. Hence, $\ddot{\imath} : \mathfrak{a} \to \hat{\mathfrak{a}}$ is given by the morphisms $\ddot{\imath} : \mathfrak{a}(X,Y) \to \widehat{\mathfrak{a}(X,Y)}$, $X, Y \in \operatorname{Ob}\mathfrak{a}$. Proposition 2.5 implies the existence of a natural transformation $\phi^n : \boxtimes_{i=1}^n \widehat{\mathfrak{a}_i} \to \widehat{\boxtimes_{i=1}^n \mathfrak{a}_i}$. According to Proposition 2.7 the equation

$$\left(\mathfrak{a}_1 \boxtimes \cdots \boxtimes \mathfrak{a}_n \xrightarrow{\ddot{\imath}_1 \boxtimes \cdots \boxtimes \ddot{\imath}_n} \hat{\mathfrak{a}}_1 \boxtimes \cdots \boxtimes \hat{\mathfrak{a}}_n \xrightarrow{\phi^n} \widehat{\mathfrak{a}_1 \boxtimes \cdots \boxtimes \mathfrak{a}_n}\right) = \ddot{\imath}_{\mathfrak{a}_1 \boxtimes \cdots \boxtimes \mathfrak{a}_n}$$

holds. Therefore, the conclusion of Proposition A.2 holds true and by Corollary A.3 we find that $\mathcal{C} = (\Lambda\text{-mod}^{\mathrm{c}}_{\mathbb{L}}\text{-}\mathsf{Quiv}, \widehat{\boxtimes}^I)$ is a symmetric monoidal $\Lambda\text{-mod}_{\mathbb{L}}$-category.

Fix a (large) set S. Consider $\Lambda\text{-mod}_{\mathbb{L}}$-category $\mathcal{D} = \Lambda\text{-mod}_{\mathbb{L}}\text{-}\mathsf{Quiv}_S$ and its reflective subcategory of complete quivers $\mathcal{C} = \Lambda\text{-mod}^{\mathrm{c}}_{\mathbb{L}}\text{-}\mathsf{Quiv}_S$. We apply the results of Appendix A to this situation as well. Again, the conclusions of Propositions 2.5 and 2.7 hold for $M_i \in \Lambda\text{-mod}_{\mathbb{L}}\text{-}\mathsf{Quiv}_S$, hence, $\mathcal{D} = (\Lambda\text{-mod}_{\mathbb{L}}\text{-}\mathsf{Quiv}_S, \otimes^I)$ and $\mathcal{C} = (\Lambda\text{-mod}^{\mathrm{c}}_{\mathbb{L}}\text{-}\mathsf{Quiv}_S, \hat{\otimes}^I)$ are monoidal $\Lambda\text{-mod}_{\mathbb{L}}$-categories. In particular, the construction of Appendix A.5.1 applies.

It will be shown later that the following simple definition is equivalent to Definition 2.40 of [De Deken, Lowen, 2018].

2.12. Definition. *Let $\mathfrak{a}$ be a complete filtered quiver and S a set. A morphism $\phi : \mathbb{1}S \to \mathfrak{a} \in \Lambda\text{-mod}^{\mathrm{c}}_{\mathbb{L}}\text{-}\mathsf{Quiv}$ is called* tensor convergent *if for every $l \in \mathbb{L}$ and every $X \in S$ there exists $N \in \mathbb{N}$ such that for every $n \geqslant N$*

$$[\phi(1_X)]^{\hat{\otimes} n} \in \mathcal{F}^l[\mathfrak{a}(\phi X, \phi X)^{\hat{\otimes} n}].$$

2.13. Lemma. *If $\phi, \psi : \mathbb{1}S \to \mathfrak{c}$ are tensor convergent and $\operatorname{Ob}\phi = \operatorname{Ob}\psi : S \to \operatorname{Ob}\mathfrak{c}$, then $\phi + \psi$ is tensor convergent as well.*

Proof. Fix an element $x \in S$ and an element $l \in \mathbb{L}$. Consider $\lambda \in \mathbb{L}_+$ such that $\lambda \geqslant l$. Denote by Y the Λ-module $\mathfrak{c}(\phi X, \phi X)$. Let $K \in \mathbb{N}$ (resp. $M \in \mathbb{N}$) be such that for every $k \in \mathbb{N}$, $k \geqslant K$ (resp. $m \in \mathbb{N}$, $m \geqslant M$) we have $\phi(1_X)^{\hat{\otimes} k} \in \mathcal{F}^{\lambda}(Y^{\hat{\otimes} k})$ (resp. $\psi(1_X)^{\hat{\otimes} m} \in \mathcal{F}^{\lambda}(Y^{\hat{\otimes} m})$). Let $\phi(1_X)^{\hat{\otimes} k} \in \mathcal{F}^{c_k}(Y^{\hat{\otimes} k})$ for $0 \leqslant k < K$ and let $c \in \mathbb{L}_+$ be such that $c + c_k \geqslant 0$ for $0 \leqslant k < K$. Let $\psi(1_X)^{\hat{\otimes} m} \in \mathcal{F}^{b_m}(Y^{\hat{\otimes} m})$ for $0 \leqslant m < M$ and let $b \in \mathbb{L}_+$ be such that $b + b_m \geqslant 0$ for $0 \leqslant m < M$. Let $N \in \mathbb{N}$ (resp. $P \in \mathbb{N}$) be such that for every $k \in \mathbb{N}$, $k \geqslant N$ (resp. $m \in \mathbb{N}$, $m \geqslant P$) we have $\phi(1_X)^{\hat{\otimes} k} \in \mathcal{F}^{\lambda+b}(Y^{\hat{\otimes} k})$ (resp. $\psi(1_X)^{\hat{\otimes} m} \in \mathcal{F}^{c+\lambda}(Y^{\hat{\otimes} m})$). Set $Q = 1 + \max\{K + P, N + M\}$. For any $n \geqslant Q$ the 2^n summands of $(\phi(1_X) + \psi(1_X))^{\hat{\otimes} n}$ are identified with one of the summands $\phi(1_X)^{\hat{\otimes} a} \hat{\otimes} \psi(1_X)^{\hat{\otimes} d}$, $a + d = n$, $a, d \in \mathbb{N}$, with the use of symmetry, preserving the filtration. If $a < K$, then $d \geqslant P$ and

$$\phi(1_X)^{\hat{\otimes} a} \hat{\otimes} \psi(1_X)^{\hat{\otimes} d} \in \mathcal{F}^{c_a}(Y^{\hat{\otimes} a}) \hat{\otimes} \mathcal{F}^{c+\lambda}(Y^{\hat{\otimes} d}) \subset \mathcal{F}^{\lambda}(Y^{\hat{\otimes} n}).$$

If $d < M$, then $a \geqslant N$ and

$$\phi(1_X)^{\hat{\otimes} a} \hat{\otimes} \psi(1_X)^{\hat{\otimes} d} \in \mathcal{F}^{\lambda+b}(Y^{\hat{\otimes} a}) \hat{\otimes} \mathcal{F}^{b_d}(Y^{\hat{\otimes} d}) \subset \mathcal{F}^{\lambda}(Y^{\hat{\otimes} n}).$$

It remains to consider the case $a \geqslant K$, $d \geqslant M$. Then

$$\phi(1_X)^{\hat{\otimes} a} \hat{\otimes} \psi(1_X)^{\hat{\otimes} d} \in \mathcal{F}^{\lambda}(Y^{\hat{\otimes} a}) \hat{\otimes} \mathcal{F}^{\lambda}(Y^{\hat{\otimes} d}) \subset \mathcal{F}^{2\lambda}(Y^{\hat{\otimes} n}).$$

Hence, $(\phi(1_X) + \psi(1_X))^{\hat{\otimes} n} \in \mathcal{F}^{\lambda}(Y^{\hat{\otimes} n}) \subset \mathcal{F}^{l}(Y^{\hat{\otimes} n})$. ∎

2.14. Remark. For any map $f : Q \to S$ and tensor convergent $\phi : \mathbb{1}S \to \mathfrak{a}$ the map $\mathbb{1}f \cdot \phi$ is tensor convergent as well. For any morphism $g : \mathfrak{a} \to \mathfrak{b} \in \Lambda\text{-mod}^{\mathrm{c}}_{\mathbb{L}}\text{-}\mathsf{Quiv}$ and any $X \in S$ the map $g^{\hat{\otimes} n} : \mathfrak{a}(\phi X, \phi X)^{\hat{\otimes} n} \to \mathfrak{b}(g\phi X, g\phi X)^{\hat{\otimes} n}$ is in $\Lambda\text{-mod}^{\mathrm{c}}_{\mathbb{L}}$, hence, $\phi \cdot g$ is tensor convergent as well.

2.15. Definition. *Inspired by Definitions 1.4, 1.6, 1.7 we say that a* completed conilpotent cocategory $\mathfrak{C} = \hat{\mathfrak{c}}$ *is a completion (as a filtered quiver) of a conilpotent cocategory $\mathfrak{c}$. It is itself a cocategory (with respect to $\hat{\otimes}$) equipped with the comultiplication $\Delta_{\hat{\mathfrak{c}}} = (\hat{\mathfrak{c}} \xrightarrow{\widehat{\Delta_{\mathfrak{c}}}} \widehat{\mathfrak{c} \otimes \mathfrak{c}} \xrightarrow{\widehat{i \otimes i}} \widehat{\hat{\mathfrak{c}} \otimes \hat{\mathfrak{c}}} = \hat{\mathfrak{c}} \hat{\otimes} \hat{\mathfrak{c}})$, the counit $\varepsilon_{\hat{\mathfrak{c}}} = \widehat{\varepsilon_{\mathfrak{c}}} : \hat{\mathfrak{c}} \to \hat{\Lambda}\hat{\mathfrak{c}} = \Lambda\mathfrak{c}$ and the augmentation $\eta_{\hat{\mathfrak{c}}} = \widehat{\eta_{\mathfrak{c}}} : \Lambda\mathfrak{c} = \hat{\Lambda}\hat{\mathfrak{c}} \to \hat{\mathfrak{c}}$, see Appendix A.5.1. Morphisms of completed conilpotent cocategories (*cofunctors*) $f : \mathfrak{B} \to \mathfrak{C}$ are morphisms of $\Lambda\text{-mod}^{\mathrm{c}}_{\mathbb{L}}\text{-}\mathsf{Quiv}$ compatible with the comultiplication and the counit in the sense of (1.2) (with $\otimes$ replaced with $\hat{\otimes}$) such that $\eta_{\mathfrak{B}} \cdot f - \mathbb{1}f \cdot \eta_{\mathfrak{C}} : \mathbb{1}\mathfrak{B} \to \mathfrak{C}$ is tensor convergent. In their set-up De Deken and Lowen introduce another notion – qA_∞-functors [De Deken, Lowen, 2018] (which turns out equivalent to cofunctors, cf. Proposition 2.28) in analogy with qdg-functors of [Polishchuk, Positselski, 2012]. The category of cofunctors between completed conilpotent cocategories is denoted* cncCat. *The category with augmentation preserving cofunctors between completed conilpotent cocategories ($\eta_{\mathfrak{B}} \cdot f = \mathbb{1}f \cdot \eta_{\mathfrak{C}}$, see (1.3)) is denoted* acncCat.

2.16. Remark. The composition of cofunctors $f : \mathfrak{a} \to \mathfrak{b}$ and $g : \mathfrak{b} \to \mathfrak{c}$ is a cofunctor itself. Indeed,

$$\begin{aligned}\eta_{\mathfrak{a}} \cdot f \cdot g - \mathbb{1}(f \cdot g) \cdot \eta_{\mathfrak{c}} &= \eta_{\mathfrak{a}} \cdot f \cdot g - \mathbb{1}f \cdot \eta_{\mathfrak{b}} \cdot g + \mathbb{1}f \cdot \eta_{\mathfrak{b}} \cdot g - \mathbb{1}f \cdot \mathbb{1}g \cdot \eta_{\mathfrak{c}} \\ &= (\eta_{\mathfrak{a}} \cdot f - \mathbb{1}f \cdot \eta_{\mathfrak{b}}) \cdot g + \mathbb{1}f \cdot (\eta_{\mathfrak{b}} \cdot g - \mathbb{1}g \cdot \eta_{\mathfrak{c}}) : \mathbb{1}\mathfrak{a} \to \mathfrak{c}.\end{aligned}$$

Both summands are tensor convergent by Remark 2.14 and induce the mapping $\operatorname{Ob} f \cdot \operatorname{Ob} g$ on objects. By Lemma 2.13 the map $\eta_{\mathfrak{a}} \cdot f \cdot g - \mathbb{1}(f \cdot g) \cdot \eta_{\mathfrak{c}}$ is tensor convergent. The identity morphism is a cofunctor. Thus, there is a category of completed conilpotent cocategories cncCat with cofunctors as morphisms.

2.17. Lemma. *Let $\phi : \mathbb{1}S \to \mathfrak{b}$, $\psi : \mathbb{1}Q \to \mathfrak{d}$ be morphisms in Λ-$\mathrm{mod}^{\mathrm{c}}_{\mathbb{L}}$-Quiv and let ϕ be tensor convergent. Then $\phi \widehat{\boxtimes} \psi : \mathbb{1}(S \times Q) = \mathbb{1}S \widehat{\boxtimes} \mathbb{1}Q \to \mathfrak{b} \widehat{\boxtimes} \mathfrak{d}$ is tensor convergent.*

Proof. Let $X \in S$, $Y \in Q$. We have $1_Y \in \mathcal{F}^0(\mathbb{1}Q(Y, Y))$, hence, $\psi(1_Y) \in \mathcal{F}^0(\mathfrak{d}(\psi Y, \psi Y))$. Therefore, $\psi(1_Y)^{\hat{\otimes} n} \in \mathcal{F}^0[\mathfrak{d}(\psi Y, \psi Y)^{\hat{\otimes} n}]$. Thus, for n large enough

$$\begin{aligned}(\phi \widehat{\boxtimes} \psi)(1_X \widehat{\boxtimes} 1_Y)]^{\hat{\otimes} n} = \phi(1_X)^{\hat{\otimes} n} \widehat{\boxtimes} \psi(1_Y)^{\hat{\otimes} n} &\in \mathcal{F}^l[\mathfrak{b}(\phi X, \phi X)^{\hat{\otimes} n}] \widehat{\boxtimes} \mathcal{F}^0[\mathfrak{d}(\psi Y, \psi Y)^{\hat{\otimes} n}] \\ &\subset \mathcal{F}^l\{[\mathfrak{b}(\phi X, \phi X) \widehat{\boxtimes} \mathfrak{d}(\psi Y, \psi Y)]^{\hat{\otimes} n}\},\end{aligned}$$

that is, $\phi \widehat{\boxtimes} \psi$ is tensor convergent. ∎

By Proposition 1.10 the category acncCat is monoidal with respect to $\widehat{\boxtimes}$. Furthermore:

2.18. Proposition. *The category* cncCat *is monoidal with respect to $\widehat{\boxtimes}$.*

Proof. First of all, the $\widehat{\boxtimes}$-product of morphisms of complete quivers, which preserve the grading and the filtration, preserves them as well. Secondly, let $f : \mathfrak{A} \to \mathfrak{B}$, $g : \mathfrak{C} \to \mathfrak{D}$ be morphisms from cncCat. Then

$$\begin{aligned}(\eta_{\mathfrak{A}} \widehat{\boxtimes} \eta_{\mathfrak{C}}) \cdot (f \widehat{\boxtimes} g) - (\mathbb{1}f \widehat{\boxtimes} \mathbb{1}g) \cdot (\eta_{\mathfrak{B}} \widehat{\boxtimes} \eta_{\mathfrak{D}}) &= \eta_{\mathfrak{A}} f \widehat{\boxtimes} \eta_{\mathfrak{C}} g - \mathbb{1}f \eta_{\mathfrak{B}} \widehat{\boxtimes} \mathbb{1}g \eta_{\mathfrak{D}} \\ &= (\eta_{\mathfrak{A}} f - \mathbb{1}f \eta_{\mathfrak{B}}) \widehat{\boxtimes} \eta_{\mathfrak{C}} g + \mathbb{1}f \eta_{\mathfrak{B}} \widehat{\boxtimes} (\eta_{\mathfrak{C}} g - \mathbb{1}g \eta_{\mathfrak{D}}) : \mathbb{1}\mathfrak{A} \widehat{\boxtimes} \mathbb{1}\mathfrak{C} \to \mathfrak{B} \widehat{\boxtimes} \mathfrak{D}.\end{aligned}$$

By Lemma 2.17 both summands are tensor convergent. Both include the mapping $\operatorname{Ob} f \times \operatorname{Ob} g$ on objects. By Lemma 2.13 their sum is tensor convergent, hence, $f \widehat{\boxtimes} g$ is a cofunctor. ∎

If cofunctor $f : \mathfrak{a} \to \mathfrak{b} \in$ ncCat, then $\hat{f} : \hat{\mathfrak{a}} \to \hat{\mathfrak{b}} \in$ acncCat. Thus, completion induces a functor ncCat $\to$ acncCat.

2.19. Example. Consider $\mathbb{L} = \{0, \infty\}$ with the neutral element 0 and the rules $0 < \infty$, $\infty + \infty = \infty$. Any abelian group M is equipped with the $\mathbb{L}$-filtration $\mathcal{F}^0 M = M$, $\mathcal{F}^\infty M = 0$. Such a filtration is called *discrete* by [De Deken, Lowen, 2018, Examples 2.4, 2.17, Remark 2.8]. The canonical mapping of $\mathbb{L}$-filtered abelian groups $\ddot{\imath} : M \to \hat{M}$ is an isomorphism. The same for graded abelian groups M. We have Λ-$\mathrm{mod}^{\mathrm{c}}_{\mathbb{L}} = \Lambda$-mod for an arbitrary graded commutative ring Λ. A morphism $\phi : \mathbb{1}S \to \mathfrak{a} \in \Lambda$-mod-Quiv

is tensor convergent if for every $X \in S$ there exists $N \in \mathbb{N}$ such that $[\phi(1_X)]^{\hat{\otimes} N} = 0$. The category cncCat has the same objects as ncCat, but larger sets of morphisms. For $f : \mathfrak{a} \to \mathfrak{b} \in$ cncCat the map $\eta_{\mathfrak{a}} \cdot f - \mathbb{1} f \cdot \eta_{\mathfrak{b}}$ is tensor convergent, while for $f \in$ ncCat it is 0.

Completion of Λ-modules commutes with direct sums in the following sense. Let M, N be filtered Λ-modules. Their direct sum is determined by the diagram $M \underset{\mathrm{in}_1}{\overset{\mathrm{pr}_1}{\leftrightarrows}} M \oplus N \underset{\mathrm{in}_2}{\overset{\mathrm{pr}_2}{\leftrightarrows}} N$ with the standard relations between pr_i and in_j. The same relations hold in the completion $\widehat{M} \underset{\widehat{\mathrm{in}_1}}{\overset{\widehat{\mathrm{pr}_1}}{\leftrightarrows}} \widehat{M \oplus N} \underset{\widehat{\mathrm{in}_2}}{\overset{\widehat{\mathrm{pr}_2}}{\leftrightarrows}} \widehat{N}$. Therefore, $\widehat{M \oplus N} \cong \widehat{M} \oplus \widehat{N}$.

Applying to the conilpotent cocategory $T\mathfrak{a}$ from Example 1.8 the completion construction of Section A.5.1, we get a functor $\widehat{T\text{-}} : \Lambda\text{-}\widetilde{\mathrm{mod}_{\mathbb{L}}}\text{-}\mathsf{Quiv} \to \mathrm{acncCat} \to \mathrm{Coalg}_{\widetilde{\Lambda\text{-}\mathrm{mod}^{\mathrm{c}}_{\mathbb{L}}}\text{-}\mathsf{Quiv}}$. The decomposition $T\mathfrak{a} = T^{<n}\mathfrak{a} \oplus T^{\geqslant n}\mathfrak{a}$, $n \geqslant 1$, implies the decomposition $\widehat{T\mathfrak{a}} = \widehat{T^{<n}\mathfrak{a}} \oplus \widehat{T^{\geqslant n}\mathfrak{a}}$.

2.20. Remark. Recall that $T\mathfrak{a}$ is also a (free) category with the composition μ, and $T^{\geqslant n}\mathfrak{a}$ is its ideal. This can be expressed as the existence of top arrow μ' in the commutative diagram

$$\begin{array}{ccc} T^{\geqslant n}\mathfrak{a} \otimes T\mathfrak{a} & \xrightarrow{\mu'} & T^{\geqslant n}\mathfrak{a} \\ {\scriptstyle i\otimes 1}\big\downarrow & & \big\downarrow{\scriptstyle i} \\ T\mathfrak{a} \otimes T\mathfrak{a} & \xrightarrow{\mu} & T\mathfrak{a} \end{array}$$

where i is the split inclusion, and by another similar diagram. Completing this square to the right square in

$$\begin{array}{ccccc} \widehat{T^{\geqslant n}\mathfrak{a}}\hat{\otimes}\widehat{T\mathfrak{a}} & \xrightarrow{\widehat{i\otimes i}^{-1}} & \widehat{T^{\geqslant n}\mathfrak{a} \otimes T\mathfrak{a}} & \xrightarrow{\widehat{\mu'}} & \widehat{T^{\geqslant n}\mathfrak{a}} \\ {\scriptstyle \widehat{i}\hat{\otimes}1}\big\downarrow & & {\scriptstyle \widehat{i\otimes 1}}\big\downarrow & & \big\downarrow{\scriptstyle \widehat{i}} \\ \widehat{T\mathfrak{a}}\hat{\otimes}\widehat{T\mathfrak{a}} & \xrightarrow{\widehat{i\otimes i}^{-1}} & \widehat{T\mathfrak{a} \otimes T\mathfrak{a}} & \xrightarrow{\widehat{\mu}} & \widehat{T\mathfrak{a}} \end{array}$$

we get a commutative diagram. Thus, $\widehat{T^{\geqslant n}\mathfrak{a}}$ is a two-sided ideal of $(\widehat{T\mathfrak{a}}, \mu_{\widehat{T\mathfrak{a}}})$.

2.21. Remark. Consider Λ-mod$_{\mathbb{L}}$-category $\mathcal{D} = \Lambda\text{-}\mathrm{mod}_{\mathbb{L}}\text{-}\mathsf{Quiv}$ and its reflective subcategory of complete quivers $\mathcal{C} = \Lambda\text{-}\mathrm{mod}^{\mathrm{c}}_{\mathbb{L}}\text{-}\mathsf{Quiv}$. Let $\mathfrak{A}$ be a completed conilpotent cocategory and let $\mathfrak{b}$ be a filtered quiver. The obvious embedding of uniform spaces $(\mathrm{pr}_k)_{k\geqslant 0} : T\mathfrak{b} \hookrightarrow \prod_{k\geqslant 0} \mathfrak{b}^{\otimes k}$, where the product is taken in $\tilde{\mathcal{D}}$, leads to embedding of completions $\widehat{T\mathfrak{b}} \subset \widehat{\prod_{k\geqslant 0} \mathfrak{b}^{\otimes k}} = \prod_{k\geqslant 0} \widehat{\mathfrak{b}^{\otimes k}}$, see [Bourbaki, 1971, Chap. II, §3, n.9, Cor. 1]. For the last equation just notice that limits commute with limits. Therefore, we have injections

$$(\text{-} \cdot \widehat{\mathrm{pr}_k})_{k\geqslant 0} : \mathrm{cncCat}(\mathfrak{A}, \widehat{T\mathfrak{b}}) \hookrightarrow \tilde{\mathcal{C}}(\mathfrak{A}, \widehat{T\mathfrak{b}}) \hookrightarrow \tilde{\mathcal{C}}(\mathfrak{A}, \prod_{k\geqslant 0} \widehat{\mathfrak{b}^{\otimes k}}) = \prod_{k\geqslant 0} \tilde{\mathcal{C}}(\mathfrak{A}, \widehat{\mathfrak{b}^{\otimes k}}), \qquad (2.6)$$

where the first product is in $\tilde{\mathcal{C}}$ and the second in $\mathcal{S}\mathrm{et}$. The completion $\widehat{T\mathfrak{b}}$ of $\oplus_{k\geqslant 0}\widehat{\mathfrak{b}^{\otimes k}} \equiv \coprod_{k\geqslant 0}\widehat{\mathfrak{b}^{\otimes k}}$ coincides with the closure of this subspace of the complete space $\prod_{k\geqslant 0}\widehat{\mathfrak{b}^{\otimes k}}$. Hence, $\widehat{T\mathfrak{b}}$ consists of elements of certain degree $x = (x_0, x_1, x_2, \dots) \in \prod_{k\geqslant 0}\widehat{\mathfrak{b}^{\otimes k}}$ such that for every $l \in \mathbb{L}$ there is $n \in \mathbb{N}$ with the property that for all $k \geqslant n$ we have $x_k \in \mathcal{F}^l\widehat{\mathfrak{b}^{\otimes k}}$. We may say also that the series $\sum_{k=0}^{\infty} x_k$ converges. This is equivalent to the previous condition since x_k belong to different direct summands.

2.22. Lemma. *Let $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$ be complete quivers. Let $f_k : \mathfrak{a} \to \mathfrak{b}$, $k \in \mathbb{N}$, and $g_m : \mathfrak{c} \to \mathfrak{d}$, $m \in \mathbb{N}$, be morphisms of filtered quivers. Assume that series $f = \sum_{k\in\mathbb{N}} f_k$ (resp. $g = \sum_{m\in\mathbb{N}} g_m$) pointwise converges, that is, for each $x \in \mathfrak{a}^d$, $d \in \mathbb{Z}$, (resp. $y \in \mathfrak{c}^p$, $p \in \mathbb{Z}$) and for every $l \in \mathbb{L}$ we have $f_k(x) \in \mathcal{F}^l\mathfrak{b}$ (resp. $g_m(y) \in \mathcal{F}^l\mathfrak{d}$) except for finite number of terms. Then the tensor product $f\widehat{\boxtimes}g : \mathfrak{a}\widehat{\boxtimes}\mathfrak{c} \to \mathfrak{b}\widehat{\boxtimes}\mathfrak{d}$ is the sum of pointwise convergent series $\sum_{k,m\in\mathbb{N}} f_k\widehat{\boxtimes}g_m$, that is, for any $x \in \mathfrak{a}^d$, $y \in \mathfrak{c}^p$, $d, p \in \mathbb{Z}$, and any $l \in \mathbb{L}$ we have $f_k(x)\widehat{\boxtimes}g_m(y) \in \mathcal{F}^l(\mathfrak{b}\widehat{\boxtimes}\mathfrak{d})$ except for finite number of terms.*

Proof. For any $l \in \mathbb{L}$ consider any decomposition $l = l' + l''$, $l', l'' \in \mathbb{L}$. Let $x \in \mathfrak{a}^d$, $y \in \mathfrak{c}^p$, $d, p \in \mathbb{Z}$. There exists $K \in \mathbb{N}$ such that $f_k(x) \in \mathcal{F}^{l'}(\mathfrak{b})$ for all $k \geqslant K$. Consider $c_k \in \mathbb{L}$ for $k < K$, $k \in \mathbb{N}$, such that $f_k(x) \in \mathcal{F}^{c_k}(\mathfrak{b})$. There is $\lambda'' \in \mathbb{L}$ such that $\lambda'' \geqslant l''$ and $c_k + \lambda'' \geqslant l$ for all $k < K$. There exists $M \in \mathbb{N}$ such that $g_m(y) \in \mathcal{F}^{\lambda''}(\mathfrak{d})$ for all $m \geqslant M$. Consider $b_m \in \mathbb{L}$ for $m < M$, $m \in \mathbb{N}$, such that $g_m(y) \in \mathcal{F}^{b_m}(\mathfrak{d})$. There is $\lambda' \in \mathbb{L}$ such that $\lambda' \geqslant l'$ and $\lambda' + b_m \geqslant l$ for all $m < M$. There exists $N \in \mathbb{N}$ such that $N \geqslant K$ and $f_k(x) \in \mathcal{F}^{\lambda'}(\mathfrak{b})$ for all $k \geqslant N$. For all pairs $(k, m) \in \mathbb{N}^2$ except such that $k < N$ and $m < M$ we deduce from the above that $f_k(x)\widehat{\boxtimes}g_m(y) \in \mathcal{F}^l(\mathfrak{b}\widehat{\boxtimes}\mathfrak{d})$. ∎

2.23. Theorem. *(i) Let $\mathfrak{A} = \hat{\mathfrak{a}}$ be a completed conilpotent cocategory and let $\mathfrak{b}$ be a filtered quiver. Cofunctors to the completed tensor cocategory $\widehat{T\mathfrak{b}}$ are in a natural bijection with the subset of quiver morphisms*

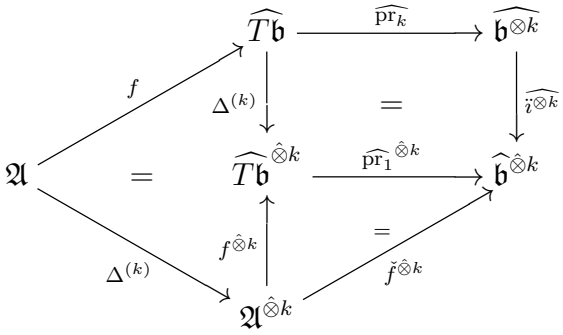
$$\mathrm{cncCat}(\mathfrak{A}, \widehat{T\mathfrak{b}}) \xrightarrow{\Phi} \{\phi : \mathfrak{A} \to \hat{\mathfrak{b}} \in \tilde{\mathcal{C}} \mid \eta\cdot\phi : \Lambda\,\mathrm{Ob}\,\mathfrak{A} \to \hat{\mathfrak{b}} \textit{ is tensor convergent}\}, \quad f \mapsto f\cdot\widehat{\mathrm{pr}_1}.$$

(ii) There is a natural bijection

$$\mathrm{acncCat}(\mathfrak{A}, \widehat{T\mathfrak{b}}) \cong \{\phi : \mathfrak{A} \to \hat{\mathfrak{b}} \in \tilde{\mathcal{C}} \mid \eta \cdot \phi = 0\}, \quad f \mapsto f \cdot \widehat{\mathrm{pr}_1}.$$

Proof. (i) First we remark that the corestriction $f \cdot \widehat{\mathrm{pr}_k}$ of any morphism $f : \mathfrak{A} \to \widehat{T\mathfrak{b}} \in \mathrm{cncCat}$ is uniquely determined by the composition $\check{f} = f \cdot \widehat{\mathrm{pr}_1} : \mathfrak{A} \to \widehat{\mathfrak{b}} \in \Lambda\text{-mod}^{\mathrm{c}}_{\mathbb{L}}\text{-}\mathsf{Quiv}$ as

the following commutative diagram shows

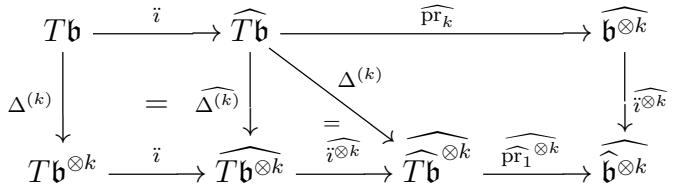


(2.7)

In fact, the obvious equation

$$\mathrm{pr}_k = \left(T\mathfrak{b} \xrightarrow{\Delta^{(k)}} T\mathfrak{b}^{\otimes k} \xrightarrow{\mathrm{pr}_1^{\otimes k}} \mathfrak{b}^{\otimes k}\right)$$

implies commutativity of the exterior of the diagram



Therefore the trapezium commute which is the upper right square in diagram (2.7).

So given $\phi : \mathfrak{A} \to \hat{\mathfrak{b}} \in \tilde{\mathcal{C}}$ such that $\eta \cdot \phi : \Lambda \operatorname{Ob} \mathfrak{A} \to \hat{\mathfrak{b}}$ is tensor convergent, let us prove that $f : \mathfrak{A} \to \widehat{T\mathfrak{b}}$ with $\operatorname{Ob} f = \operatorname{Ob} \phi$ and for any element $x \in \mathfrak{A}^d$, $d \in \mathbb{Z}$, the value of $f(x)$ given by the (convergent) series

$$\sum_{k \geqslant 0} \widehat{\ddot{\imath}^{\otimes k}}^{-1} \phi^{\hat{\otimes} k}(\Delta_{\mathfrak{A}}^{(k)} x) = \sum_{k \geqslant 0} \hat{\imath}^{-1}\phi(x_{(1)}) \hat{\otimes} \cdots \hat{\otimes} \hat{\imath}^{-1}\phi(x_{(k)})$$
$$= (x)\varepsilon \cdot (\mathbb{1}\phi) \cdot \mathrm{in}_0 + \sum_{k \geqslant 1} \hat{\imath}^{-1}\phi(x_{(1)}) \hat{\otimes} \cdots \hat{\otimes} \hat{\imath}^{-1}\phi(x_{(k)}), \quad (2.8)$$

is a cofunctor. Here $x_{(1)} \hat{\otimes} \cdots \hat{\otimes} x_{(k)} \equiv \Delta^{(k)}(x)$. Convergence means that for every $l \in \mathbb{L}$ the k-th term except for a finite number of terms belongs to $\mathcal{F}^l \widehat{\mathfrak{b}^{\otimes k}}$ and will be proven now.

Assume that $\mathfrak{A} = \hat{\mathfrak{a}}$ where the cocategory $\mathfrak{a}$ is conilpotent and $\phi : \mathfrak{A} \to \hat{\mathfrak{b}} \in \tilde{\mathcal{C}}$ is such that $\eta_{\mathfrak{A}} \cdot \phi$ tensor converges. Replacing $\mathfrak{a}$ with the conilpotent cocategory $\ddot{\imath}(\mathfrak{a})$ we may assume that $\ddot{\imath} : \mathfrak{a} \hookrightarrow \hat{\mathfrak{a}}$ is an embedding. We have to prove that (2.8) converges for all $x \in \mathfrak{A}^d$. It suffices to assume that $x \in \bar{\mathfrak{A}}(X, Y)^d$. Use the notation $x_{(\bar{1})} \hat{\otimes} \cdots \hat{\otimes} x_{(\bar{k})} \equiv \bar{\Delta}_{\mathfrak{A}}^{(k)}(x)$. The counital comultiplication Δ is recovered from the reduced comultiplication $\bar{\Delta}$ via the formula

$$\Delta_{\mathfrak{A}}^{(k)}(x) = \sum_{j : S \hookrightarrow \mathbf{k}}^{S \neq \varnothing} \hat{\otimes}^{i \in \mathbf{k}} (x_{\overline{(j^{-1}i)}})^{\chi(i \in jS)}, \quad (2.9)$$

where $y^0 = \eta_{\mathfrak{A}}(1)$, $y^1 = y$, and the summation extends over all non-empty subsets S of $\mathbf{k} = \{1, 2, \dots, k\}$. By convention, $\bar{\Delta}^{(1)}_{\mathfrak{A}} x = x$.

Assume for a moment that $x \in \bar{\mathfrak{a}}^d$, hence, for some $n > 0$ $\bar{\Delta}^{(n)}_{\mathfrak{a}}(x) = 0$. So the summation in (2.9) goes over S with $|S| < n$. The list of tuples of objects $X = Z_0, Z_1, \dots, Z_{m-1}, Z_m = Y$, which occur in

$$\bar{\Delta}^{(m)}_{\mathfrak{a}}(x) \in \oplus_{Z_1,\dots,Z_{m-1}} \mathfrak{a}(X, Z_1) \otimes \mathfrak{a}(Z_1, Z_2) \otimes \dots \otimes \mathfrak{a}(Z_{m-1}, Y)$$

for $1 \leqslant m < n$, is finite. Form a single list $Z_1, \dots, Z_q$ of Z_i occurring in all such decompositions and denote $Y_q = \hat{\mathfrak{b}}(\phi Z_q, \phi Z_q) \in \Lambda\text{-mod}^{\mathrm{c}}_{\mathbb{L}}$. Let $P \in \mathbb{N}$ such that for every $p \in \mathbb{N}$, $p \geqslant P$, we have $\phi\eta(1_{Z_q})^{\otimes p} \in \mathcal{F}^0(Y_q^{\otimes p})$. Let $b_p \in \mathbb{L}$ be such that $\phi\eta(1_{Z_q})^{\otimes p} \in \mathcal{F}^{b_p}(Y_q^{\otimes p})$ for any $1 \leqslant q \leqslant Q$ and any $p < P$. Let $b \leqslant b_p$ for all $p < P$. Let $c \in \mathbb{L}$ be such that all factors $x_{(\bar{i})}$ of all summands of all $\Delta^{(m)}(x)$, $0 \leqslant m < n$, belonging to $\mathfrak{a}(Z_{i-1}, Z_i)$, were, in fact, in $\mathcal{F}^c\mathfrak{a}(Z_{i-1}, Z_i)$. There is $\lambda \in \mathbb{L}$ such that $(n-1)c + nb + \lambda \geqslant l$. There is $N \in \mathbb{N}$ such that $k \geqslant N$ implies $\phi\eta(1_{Z_q})^{\otimes k} \in \mathcal{F}^{\lambda}(Y_q^{\otimes k})$. When $k > nN$ at least one factor of this type occurs in $\Delta^{(k)}(x)$. Hence, for $k > nN$ we have $\phi^{\otimes k}\Delta^{(k)}(x) \in \mathcal{F}^{\lambda}(\hat{\mathfrak{b}}(\phi X, \phi Y))$.

Consider now an arbitrary $x \in \bar{\mathfrak{A}}(X, Y)^d$. Given $l \in \mathbb{L}$ there is an element $x' \in \bar{\mathfrak{a}}(X, Y)^d$ such that $x - x' \in \mathcal{F}^l\bar{\mathfrak{A}}(X, Y)^d$. Then for all $k \geqslant 1$ we have $\phi^{\hat{\otimes} k}(\Delta^{(k)}_{\mathfrak{A}} x) - \ddot{\imath}[\phi^{\otimes k}(\Delta^{(k)}_{\mathfrak{a}} x')] \in \mathcal{F}^l(\hat{\mathfrak{b}}^{\hat{\otimes} k})$. Since $\phi^{\otimes k}(\Delta^{(k)}_{\mathfrak{a}} x') \in \mathcal{F}^l(\hat{\mathfrak{b}}^{\otimes k})$ for large k we deduce that $\phi^{\hat{\otimes} k}(\Delta^{(k)}_{\mathfrak{A}} x) \in \mathcal{F}^l(\hat{\mathfrak{b}}^{\hat{\otimes} k})$ which proves the convergence of (2.8) and gives a well-defined map of filtered quivers $f : \mathfrak{A} \to \widehat{T\mathfrak{b}}$ with $\operatorname{Ob} f = \operatorname{Ob} \phi$.

Let us prove that f is a morphism of cocategories. Due to coassociativity of $\Delta_{\mathfrak{A}}$ we may write a convergent series

$$(x)f \cdot \Delta = \sum_{k \in \mathbb{N}} \sum_{m+n=k}^{m,n \in \mathbb{N}} \phi^{\hat{\otimes} m}(\Delta^{(m)} x_{(1)}) \hat{\otimes} \phi^{\hat{\otimes} n}(\Delta^{(n)} x_{(2)}). \tag{2.10}$$

Variant of Lemma 2.22 for $\hat{\otimes}$ gives another convergent series

$$(x)\Delta \cdot (f \hat{\otimes} f) = \sum_{m,n \in \mathbb{N}} \phi^{\hat{\otimes} m}(\Delta^{(m)} x_{(1)}) \hat{\otimes} \phi^{\hat{\otimes} n}(\Delta^{(n)} x_{(2)})$$

with the same terms as in (2.10) but with different summation order. Since the sum of a series convergent in our sense does not depend on the order of summation, we conclude that $f \cdot \Delta = \Delta \cdot (f \hat{\otimes} f)$.

The morphism f preserves the counit due to (2.8):

$$f \cdot \varepsilon = f \cdot \widehat{\mathrm{pr}_0} = \varepsilon \cdot (\mathbb{1}\phi) = \varepsilon \cdot (\mathbb{1}f).$$

The map f is a cofunctor since

$$(1_X)\eta_{\mathfrak{A}} \cdot f - (1_X)\mathbb{1}f \cdot \mathrm{in}_0 \cdot \ddot{\imath} = \sum_{k \geqslant 1} [\phi\eta(1_X)]^{\hat{\otimes} k}.$$

By definition, for any $l \in \mathbb{L}_+$ there exists $N \in \mathbb{N}$ such that for every $n \geqslant N$ we have $[\phi\eta(1_X)]^{\hat{\otimes} n} \in \mathcal{F}^l[\widehat{\mathfrak{b}^{\otimes n}}]$. Clearly, all terms of

$$[(1_X)\eta_{\mathfrak{A}} \cdot f - (1_X)\mathbb{1}f \cdot \mathrm{in}_0 \cdot \ddot{\imath}]^{\hat{\otimes} n} = \Big[\sum_{k \geqslant 1} [\phi\eta(1_X)]^{\hat{\otimes} k}\Big]^{\hat{\otimes} n}$$

are in $\mathcal{F}^l[\widehat{T\mathfrak{b}}]$. Summing up, a map

$$\Psi : \{\phi : \mathfrak{A} \to \hat{\mathfrak{b}} \in \tilde{\mathcal{C}} \mid \eta \cdot \phi : \Lambda \operatorname{Ob} \mathfrak{A} \to \hat{\mathfrak{b}} \text{ is tensor convergent}\} \to \mathrm{cncCat}(\mathfrak{A}, \widehat{T\mathfrak{b}})$$

is constructed.

Let us prove that for any cofunctor $f : \mathfrak{A} \to \widehat{T\mathfrak{b}}$ the map $\eta \cdot \check{f} = \eta \cdot f \cdot \widehat{\mathrm{pr}_1} : \mathbb{1}\mathfrak{A} \to \widehat{\mathfrak{b}}$ is tensor convergent. We know that $\eta_{\mathfrak{A}} \cdot f - \mathbb{1}f \cdot \mathrm{in}_0 \cdot \ddot{\imath} : \mathbb{1}\mathfrak{A} \to \widehat{T\mathfrak{b}}$ is tensor convergent. Therefore, $(\eta_{\mathfrak{A}} \cdot f - \mathbb{1}f \cdot \mathrm{in}_0 \cdot \ddot{\imath}) \cdot \widehat{\mathrm{pr}_1} = \eta \cdot \check{f}$ is tensor convergent by Remark 2.14. Thus, the claimed map $\Phi : f \mapsto \check{f}$ is constructed.

Clearly, $\Phi\Psi(\phi) = \phi$. In particular, Φ is surjective. As the reasoning at the beginning of the proof shows, injection (2.6) factorizes through Φ, namely, $(\text{-} \cdot \widehat{\mathrm{pr}_k})_{k \geqslant 0} = \Phi \cdot \Xi$, where

$$\Xi : \tilde{\mathcal{C}}(\mathfrak{A}, \widehat{\mathfrak{b}}) \hookrightarrow \prod_{k \geqslant 0} \tilde{\mathcal{C}}(\mathfrak{A}, \widehat{\mathfrak{b}^{\otimes k}}), \qquad \phi \mapsto (\Delta_{\mathfrak{A}}^{(k)} \cdot \phi^{\hat{\otimes} k})_{k \geqslant 0}.$$

Therefore, Φ is an injection as well. We conclude that Φ is bijective and $\Psi = \Phi^{-1}$.

(ii) follows from (i). ∎

2.24. Corollary. *Let $\mathfrak{a}$ be a conilpotent cocategory and let $\mathfrak{b}$ be a filtered quiver.*

(i) Cofunctors to the completed tensor cocategory $\widehat{T\mathfrak{b}}$ are in a natural bijection with the subset of quiver morphisms

$$\mathrm{cncCat}(\hat{\mathfrak{a}}, \widehat{T\mathfrak{b}}) \cong \{\phi : \mathfrak{a} \to \hat{\mathfrak{b}} \in \tilde{\mathcal{C}} \mid \eta \cdot \phi : \Lambda \operatorname{Ob} \mathfrak{a} \to \hat{\mathfrak{b}} \textit{ is tensor convergent}\}, \quad f \mapsto f \cdot \widehat{\mathrm{pr}_1}.$$

(ii) There is a natural bijection

$$\mathrm{acncCat}(\hat{\mathfrak{a}}, \widehat{T\mathfrak{b}}) \cong \{\phi : \mathfrak{a} \to \hat{\mathfrak{b}} \in \tilde{\mathcal{C}} \mid \eta \cdot \phi = 0\}, \quad f \mapsto f \cdot \widehat{\mathrm{pr}_1}.$$

Proof. Follows from Proposition 2.1(i) and Theorem 2.23 by universality property of the completion. ∎

2.25. Definition. *Let $\mathfrak{a} \in \mathrm{ncCat}$, $\mathfrak{B} \in \mathrm{cncCat}$. A* cofunctor $f : \mathfrak{a} \to \mathfrak{B}$ *is a morphism from Λ-mod$_{\mathbb{L}}$-*Quiv *compatible with the comultiplication and the counit in the sense that*

$$\begin{array}{ccc} \mathfrak{a} & \xrightarrow{\quad f \quad} & \mathfrak{B} \\ {\scriptstyle\Delta}\downarrow & = & \downarrow{\scriptstyle\Delta} \\ \mathfrak{a} \otimes \mathfrak{a} & \xrightarrow{f \otimes f} \mathfrak{B} \otimes \mathfrak{B} & \xrightarrow{\ddot{\imath}} \widehat{\mathfrak{B} \otimes \mathfrak{B}} \end{array} \quad , \quad \begin{array}{ccc} \mathfrak{a} & \xrightarrow{\quad f \quad} & \mathfrak{B} \\ {\scriptstyle\varepsilon}\downarrow & = & \downarrow{\scriptstyle\varepsilon} \\ \Lambda\mathfrak{a} & \xrightarrow{\Lambda f} & \Lambda\mathfrak{B} \end{array} \tag{2.11}$$

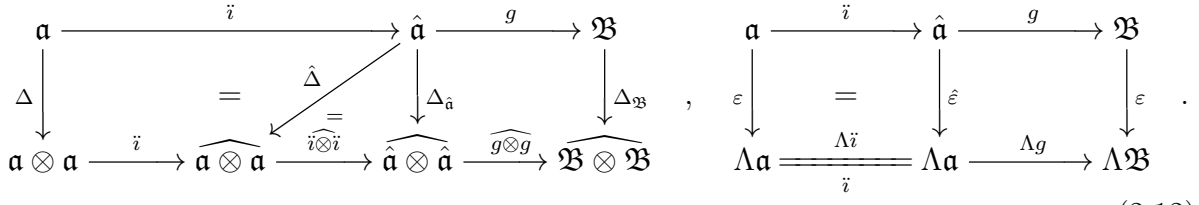
and such that $\eta_{\mathfrak{a}} \cdot f - \mathbb{1}f \cdot \eta_{\mathfrak{B}}$ is tensor convergent.

An explanation of the above is given by

2.26. Proposition. *Let $\mathfrak{a} \in \mathrm{ncCat}$, $\mathfrak{B} \in \mathrm{cncCat}$. The restriction and universality of the completion give mutually inverse bijections*

$$\mathrm{cncCat}(\hat{\mathfrak{a}}, \mathfrak{B}) \longleftrightarrow \{ \textit{ cofunctors } \mathfrak{a} \to \mathfrak{B}\}.$$

Proof. Diagrams (2.11) for $f = \ddot{\imath} : \mathfrak{a} \to \hat{\mathfrak{a}}$ take the form of left rectangles below



(2.12)

and they obviously commute, proving that $\ddot{\imath} : \mathfrak{a} \to \hat{\mathfrak{a}}$ is a cofunctor. The restriction map is $\mathrm{cncCat}(\hat{\mathfrak{a}}, \mathfrak{B}) \to \{$ cofunctors $\mathfrak{a} \to \mathfrak{B}\}$, $g \mapsto \ddot{\imath} \cdot g$.

The inverse map is constructed as follows. Let $f : \mathfrak{a} \to \mathfrak{B} \in \Lambda\text{-mod}_{\mathbb{L}}\text{-}\mathsf{Quiv}$ satisfy (2.11). Then there is a unique $g = \tilde{f} : \hat{\mathfrak{a}} \to \mathfrak{B} \in \Lambda\text{-mod}_{\mathbb{L}}\text{-}\mathsf{Quiv}$ such that $f = \big(\mathfrak{a} \xrightarrow{\ddot{\imath}} \hat{\mathfrak{a}} \xrightarrow{g} \mathfrak{B}\big)$. If f is a cofunctor, the exterior rectangles of (2.12) commute. By universality property of $\ddot{\imath} : \mathfrak{a} \to \hat{\mathfrak{a}}$, the right squares of (2.12) commute as well. ∎

Introduce the notation $\hat{T}\mathfrak{a} = \widehat{T\mathfrak{a}}$. We have shown in the above proof that any cofunctor $f : T\mathfrak{a} \to \hat{T}\mathfrak{b}$ factorizes as $f = \big(T\mathfrak{a} \xrightarrow{\ddot{\imath}} \hat{T}\mathfrak{a} \xrightarrow{\tilde{f}} \hat{T}\mathfrak{b}\big)$ for a unique $\tilde{f} \in \mathrm{cncCat}(\hat{T}\mathfrak{a}, \hat{T}\mathfrak{b})$. The components of f and $\tilde{f}$,

$$f_k = \big(T^k\mathfrak{a} \xrightarrow{\mathrm{in}_k} T\mathfrak{a} \xrightarrow{f} \hat{T}\mathfrak{b} \xrightarrow{\widehat{\mathrm{pr}_1}} \hat{\mathfrak{b}}\big),$$
$$\tilde{f}_k = \big(\widehat{T^k\mathfrak{a}} \xrightarrow{\widehat{\mathrm{in}_k}} \hat{T}\mathfrak{a} \xrightarrow{\tilde{f}} \hat{T}\mathfrak{b} \xrightarrow{\widehat{\mathrm{pr}_1}} \hat{\mathfrak{b}}\big),$$

are related by $f_k = \ddot{\imath} \cdot \tilde{f}_k$ as well.

2.27. Remark. It follows from Remark 2.21 that the series

$$\sum_{k=0}^{\infty} \widehat{\mathrm{pr}_k} \cdot \widehat{\mathrm{in}_k} = \sum_{k=0}^{\infty} \big(\hat{T}\mathfrak{b} \xrightarrow{\widehat{\mathrm{pr}_k}} \widehat{T^k\mathfrak{b}} \xrightarrow{\widehat{\mathrm{in}_k}} \hat{T}\mathfrak{b}\big)$$

converges to $\mathrm{Id}_{\hat{T}\mathfrak{b}}$.

We use this remark in order to write down components of the composition $h = \big(T\mathfrak{a} \xrightarrow{f} \hat{T}\mathfrak{b} \xrightarrow{\tilde{g}} \hat{T}\mathfrak{c}\big)$. We have by (2.7)

$$\begin{aligned}
h_l &= \sum_{k=0}^{\infty} \big(T^l\mathfrak{a} \xrightarrow{\mathrm{in}_l} T\mathfrak{a} \xrightarrow{f} \hat{T}\mathfrak{b} \xrightarrow{\widehat{\mathrm{pr}_k}} \widehat{T^k\mathfrak{b}} \xrightarrow{\widehat{\mathrm{in}_k}} \hat{T}\mathfrak{b} \xrightarrow{\tilde{g}} \hat{T}\mathfrak{c} \xrightarrow{\widehat{\mathrm{pr}_1}} \hat{\mathfrak{c}}\big) \\
&= \sum_{k=0}^{\infty} \big(T^l\mathfrak{a} \xrightarrow{\mathrm{in}_l} T\mathfrak{a} \xrightarrow{\Delta^{(k)}} (T\mathfrak{a})^{\otimes k} \xrightarrow{\check{f}^{\otimes k}} \hat{\mathfrak{b}}^{\otimes k} \xrightarrow{\ddot{\imath}} \hat{\mathfrak{b}}^{\widehat{\otimes} k} \xrightarrow{\widehat{\ddot{\imath}^{\otimes k}}^{-1}} \mathfrak{b}^{\widehat{\otimes} k} \xrightarrow{\tilde{g}_k} \hat{\mathfrak{c}}\big) \\
&= \sum_{i_1+\cdots+i_k=l}^{k\geqslant 0} \big(T^l\mathfrak{a} \xrightarrow{f_{i_1}\otimes\cdots\otimes f_{i_k}} \hat{\mathfrak{b}}^{\otimes k} \xrightarrow{\ddot{\imath}} \hat{\mathfrak{b}}^{\widehat{\otimes} k} \xrightarrow{\widehat{\ddot{\imath}^{\otimes k}}^{-1}} \mathfrak{b}^{\widehat{\otimes} k} \xrightarrow{\tilde{g}_k} \hat{\mathfrak{c}}\big).
\end{aligned} \tag{2.13}$$

2.28. Proposition. *Let $\mathfrak{a}$, $\mathfrak{b}$ be filtered quivers and let $f : \hat{T}\mathfrak{a} \to \hat{T}\mathfrak{b} \in \Lambda\text{-mod}^{\mathrm{c}}_{\mathbb{L}}\text{-}\mathsf{Quiv}$ be compatible with the comultiplication and the counit. Then f is a cofunctor iff $f_0 = \widehat{\mathrm{in}_0} \cdot f \cdot \widehat{\mathrm{pr}_1} : \mathbb{1}\mathfrak{a} \to \hat{\mathfrak{b}}$ is tensor convergent.*

Proof. Assuming that $f_0 : \mathbb{1}\mathfrak{a} \to \hat{\mathfrak{b}}$ is tensor convergent, we find due to diagram (2.7) that $\eta_{\hat{T}\mathfrak{a}} \cdot f - \mathbb{1}f \cdot \eta_{\hat{T}\mathfrak{b}} = \sum_{n>0} f_0^{\hat{\otimes} n} : \mathbb{1}\mathfrak{a} \to \hat{T}\mathfrak{b}$ and the series in the right hand side is convergent. Furthermore, the series in the right hand side is tensor convergent. Therefore, if f_0 is tensor convergent, then f is cofunctor.

Assuming that f is a cofunctor, we see that by definition $y = \eta_{\hat{T}\mathfrak{a}} \cdot f - \mathbb{1}f \cdot \eta_{\hat{T}\mathfrak{b}} : \mathbb{1}\mathfrak{a} \to \widehat{T^{\geqslant 1}\mathfrak{b}}$ is tensor convergent. From (2.8) we obtain that $y = \sum_{n>0} f_0^{\hat{\otimes} n}$. The series $f_0' = \sum_{m>0} (-1)^{m-1} y^{\hat{\otimes} m} : \mathbb{1}\mathfrak{a} \to \widehat{T^{\geqslant 1}\mathfrak{b}}$ converges. Let us compute the sum:

$$f_0' = \sum_{m>0} (-1)^{m-1} \Big(\sum_{n>0} f_0^{\hat{\otimes} n} \Big)^{\hat{\otimes} m} = \sum_{k>0} f_0^{\hat{\otimes} k} \sum_{i_1 + \cdots + i_m = k}^{m, i_j > 0} (-1)^{m-1}.$$

Notice that the coefficient near t^k in expansion $(\frac{t}{1-t})^m = t^m \sum_{a=0}^{\infty} (-1)^a t^a \binom{-m}{a}$ equals $\binom{k-1}{k-m}$ for $k \geqslant m$ and vanishes if $k < m$. Therefore,

$$f_0' = \sum_{k>0} f_0^{\hat{\otimes} k} \sum_{m=1}^{k} (-1)^{m-1} \binom{k-1}{k-m} = f_0 + \sum_{k>1} f_0^{\hat{\otimes} k} (1-1)^{k-1} = f_0.$$

We conclude that $f_0 = f_0'$ is tensor convergent. By the way, one can show that the both compositions of the maps $f_0 \mapsto y$, $y \mapsto f_0$ are identities. ∎

This proposition shows that in the set-up of De Deken and Lowen qA_∞-functors [De Deken, Lowen, 2018] are the same as cofunctors.

2.29. Corollary. *Let $\mathfrak{a}$, $\mathfrak{b}$ be filtered quivers and let $f : T\mathfrak{a} \to \hat{T}\mathfrak{b} \in \Lambda\text{-mod}_{\mathbb{L}}\text{-}\mathsf{Quiv}$ be compatible with the comultiplication and the counit in the sense of diagrams (2.11). Then f is a cofunctor iff $f_0 = \mathrm{in}_0 \cdot f \cdot \widehat{\mathrm{pr}_1} : \mathbb{1}\mathfrak{a} \to \hat{\mathfrak{b}}$ is tensor convergent.*

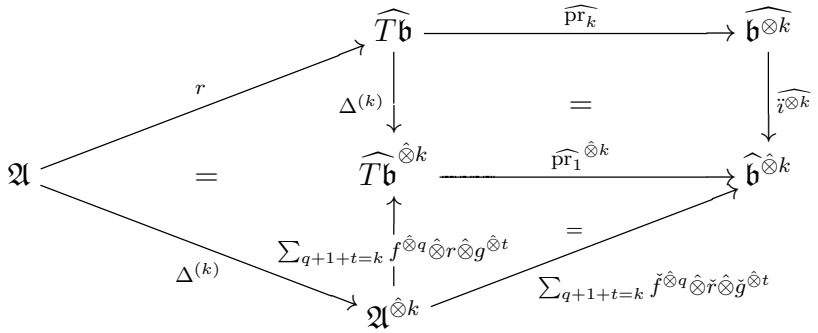
2.30. Coderivations.

2.31. Definition. *Let $f, g : \mathfrak{A} \to \mathfrak{B} \in \mathrm{cncCat}$. An (f,g)-coderivation $r : f \to g : \mathfrak{A} \to \mathfrak{B}$ of degree d and of level λ is a collection of elements $r \in \mathcal{F}^\lambda \underline{\Lambda\text{-mod}_{\mathbb{L}}}(\mathfrak{A}(X,Y), \mathfrak{B}(fX, gY))^d$, which satisfies the equation $r \cdot \Delta = \Delta \cdot (f \hat{\otimes} r + r \hat{\otimes} g)$.*

Let $\mathfrak{A}, \mathfrak{B} \in \mathrm{cncCat}$. The coderivation quiver $\mathrm{Coder}(\mathfrak{A}, \mathfrak{B})$ has cofunctors $f : \mathfrak{A} \to \mathfrak{B}$ as objects and the component $\mathcal{F}^\lambda \mathrm{Coder}(\mathfrak{A}, \mathfrak{B})(f,g)^d$ of the filtered graded Λ-module $\mathrm{Coder}(\mathfrak{A}, \mathfrak{B})(f,g)$ consists of coderivations $r : f \to g : \mathfrak{A} \to \mathfrak{B}$ of degree d and of level λ.

2.32. Proposition. *Let $\mathfrak{b}$ be a filtered quiver and let $f, g : \mathfrak{A} \to \widehat{T\mathfrak{b}} \in \mathrm{cncCat}$. (f,g)-coderivations $r : f \to g : \mathfrak{A} \to \widehat{T\mathfrak{b}}$ of degree d and of level λ are in bijection with the collections of morphisms $\check{r} = r \cdot \widehat{\mathrm{pr}_1} \in \mathcal{F}^\lambda \underline{\Lambda\text{-mod}_{\mathbb{L}}}(\mathfrak{A}(X,Y), \hat{\mathfrak{b}}(fX, gY))^d$, $X, Y \in \mathrm{Ob}\,\mathfrak{A}$.*

Proof. The commutative diagram



shows that the composition $\mathfrak{A} \xrightarrow{r} \widehat{T\mathfrak{b}} \hookrightarrow \prod_{k\geqslant 0} \widehat{\mathfrak{b}^{\otimes k}}$ is given by the family

$$\big(\Delta^{(k)}_{\mathfrak{A}} \cdot \sum_{q+1+t=k} \check{f}^{\hat{\otimes} q}\hat{\otimes}\check{r}\hat{\otimes}\check{g}^{\hat{\otimes} t}\big)_{k=0}^{\infty}.$$

Due to coassociativity of $\Delta_{\mathfrak{A}}$ this equals $\Delta^{(3)}_{\mathfrak{A}} \cdot (f\hat{\otimes}\check{r}\hat{\otimes} g) \cdot \mu^{(3)}_{\widehat{T\mathfrak{b}}}$, which clearly lies in $\widehat{T\mathfrak{b}}$. Thus,

$$r = \Delta^{(3)}_{\mathfrak{A}} \cdot (f\hat{\otimes}\check{r}\hat{\otimes} g) \cdot \mu^{(3)}_{\widehat{T\mathfrak{b}}} \tag{2.14}$$

is unambiguously determined by the collection $\check{r} = r \cdot \widehat{\mathrm{pr}_1} : \mathfrak{A}(X,Y) \to \hat{\mathfrak{b}}(fX, gY)$.

On the other hand, the right hand side of (2.14) is an (f,g)-coderivation as the following computation shows

$$\begin{aligned} x(r\cdot\Delta) &= (x_{(1)}\hat{\otimes}x_{(2)}\hat{\otimes}x_{(3)})[(f\cdot\Delta)\hat{\otimes}\check{r}\hat{\otimes}g] + (x_{(1)}\hat{\otimes}x_{(2)}\hat{\otimes}x_{(3)})[f\hat{\otimes}\check{r}\hat{\otimes}(g\cdot\Delta)] \\ &= \big(x_{(1)}\widehat{\bigotimes}x_{(2)}\hat{\otimes}x_{(3)}\hat{\otimes}x_{(4)}\big)\big(f\widehat{\bigotimes}f\hat{\otimes}\check{r}\hat{\otimes}g\big) + \big(x_{(1)}\hat{\otimes}x_{(2)}\hat{\otimes}x_{(3)}\widehat{\bigotimes}x_{(4)}\big)\big(f\hat{\otimes}\check{r}\hat{\otimes}g\widehat{\bigotimes}g\big) \\ &= \big(x_{(1)}\widehat{\bigotimes}x_{(2)}\big)\big(f\widehat{\bigotimes}r\big) + \big(x_{(1)}\widehat{\bigotimes}x_{(2)}\big)\big(r\widehat{\bigotimes}g\big) = (x\Delta)(f\hat{\otimes}r + r\hat{\otimes}g). \end{aligned}$$

It remains to note that f and g preserve the filtration and the grading. ∎

2.33. Corollary. *Let $\mathfrak{a} \in$ ncCat, let $\mathfrak{b}$ be a filtered quiver and let $f, g : \hat{\mathfrak{a}} \to \widehat{T\mathfrak{b}} \in$ cncCat. (f,g)-coderivations $r : f \to g : \hat{\mathfrak{a}} \to \widehat{T\mathfrak{b}}$ of degree d and of level λ are in bijection with the collections of morphisms $\check{r} = r \cdot \widehat{\mathrm{pr}_1} \in \mathcal{F}^{\lambda}\underline{\Lambda\text{-mod}}_{\mathbb{L}}(\mathfrak{a}(X,Y), \hat{\mathfrak{b}}(fX,gY))^d$, $X, Y \in \operatorname{Ob}\mathfrak{a}$.*

Proof. Follows from Propositions 2.1(i) and 2.32 by universality property of the completion. ∎

2.34. Definition. *Let $\mathfrak{a} \in$ ncCat, $\mathfrak{B} \in$ cncCat, and let $f, g : \mathfrak{a} \to \mathfrak{B}$ be cofunctors in the sense of Definition 2.25. An (f,g)-coderivation $r : f \to g : \mathfrak{a} \to \mathfrak{B}$ of degree d and of level λ is a collection of elements $r \in \mathcal{F}^{\lambda}\underline{\Lambda\text{-mod}}_{\mathbb{L}}(\mathfrak{a}(X,Y), \mathfrak{B}(fX,gY))^d$, which satisfies*

the equation

$$\begin{array}{ccccc} \mathfrak{a} & & \xrightarrow{\quad r \quad} & & \mathfrak{B} \\ {\scriptstyle\Delta}\downarrow & & = & & \downarrow{\scriptstyle\Delta} \\ \mathfrak{a}\otimes\mathfrak{a} & \xrightarrow{f\otimes r+r\otimes g} & \mathfrak{B}\otimes\mathfrak{B} & \xrightarrow{\ddot{\imath}} & \widehat{\mathfrak{B}\otimes\mathfrak{B}} \end{array} \tag{2.15}$$

The filtered Λ*-module of* (f,g)*-coderivations is denoted* $\operatorname{Coder}(\mathfrak{a},\mathfrak{B})(f,g)$.

The reason for introducing this definition is given by the following

2.35. Proposition. *Let* $\mathfrak{a}\in\mathrm{ncCat}$, $\mathfrak{B}\in\mathrm{cncCat}$, *and let* $f,g:\mathfrak{a}\to\mathfrak{B}$ *be cofunctors. They can be represented as* $f=\ddot{\imath}\cdot f'$, $g=\ddot{\imath}\cdot g'$ *by Proposition 2.26. Then the map*

$$\operatorname{Coder}(\hat{\mathfrak{a}},\mathfrak{B})(f',g')\to\operatorname{Coder}(\mathfrak{a},\mathfrak{B})(f,g),\qquad r'\mapsto\ddot{\imath}\cdot r'=r, \tag{2.16}$$

is a bijection.

Proof. Take $r'\in\operatorname{Coder}(\hat{\mathfrak{a}},\mathfrak{B})(f',g')$. Then the rightmost quadrilateral (trapezium) in the following diagram commutes:

$$\begin{array}{ccccccc} \mathfrak{a} & \xrightarrow{\ddot{\imath}} & \hat{\mathfrak{a}} & & \xrightarrow{\quad r' \quad} & & \mathfrak{B} \\ {\scriptstyle\Delta}\downarrow & = & \hat{\Delta}\downarrow & \searrow^{\Delta_{\hat{\mathfrak{a}}}} & & & \downarrow{\scriptstyle\Delta} \\ \mathfrak{a}\otimes\mathfrak{a} & \xrightarrow{\ddot{\imath}} & \widehat{\mathfrak{a}\otimes\mathfrak{a}} & \xrightarrow[\ddot{\imath}\widehat{\otimes}\ddot{\imath}]{=} & \hat{\mathfrak{a}}\otimes\hat{\mathfrak{a}} & \xrightarrow{\widehat{f'\otimes r'+r'\otimes g'}} & \widehat{\mathfrak{B}\otimes\mathfrak{B}} \\ & {\scriptstyle f\otimes r+r\otimes g}\searrow & & = & & \nearrow{\scriptstyle\ddot{\imath}} & \\ & & & \mathfrak{B}\otimes\mathfrak{B} & & & \end{array} \tag{2.17}$$

Therefore, the whole diagram commutes and map (2.16) is well-defined.

On the other hand, any map $r\in\mathcal{F}^\lambda\underline{\Lambda\text{-mod}_{\mathbb{L}}}(M,N)^d$ takes $\mathcal{F}^lM^k$ to $\mathcal{F}^{l+\lambda}N^{k+d}$. We are interested in $M=\mathfrak{a}(X,Y)$, $N=\mathfrak{B}(fX,gY)$. By Proposition 2.1(i) $r:M^k\to N^{k+d}$ are uniformly continuous for all $k\in\mathbb{Z}$. Therefore, these maps factorize as $r=\big(M^k\xrightarrow{\ddot{\imath}}\widehat{M^k}\xrightarrow{r'}N^{k+d}\big)$. Clearly, $r'\in\mathcal{F}^\lambda\underline{\Lambda\text{-mod}_{\mathbb{L}}}(\hat{M},N)^d$ and $r=\ddot{\imath}\cdot r'$. The exterior of (2.17) commutes. Thus, the biggest rectangle in (2.17) commutes. Hence, the right rectangle commutes. Equivalently, the trapezium with vertices $\hat{\mathfrak{a}},\dots,\widehat{\mathfrak{B}\otimes\mathfrak{B}}$ commutes, that is, r' is an (f',g')-coderivation. ∎

2.36. Corollary. *Let* $\mathfrak{a}\in\mathrm{ncCat}$, $\mathfrak{B}\in\mathrm{cncCat}$. *Then the filtered quivers* $\operatorname{Coder}(\hat{\mathfrak{a}},\mathfrak{B})$ *and* $\operatorname{Coder}(\mathfrak{a},\mathfrak{B})$ *are isomorphic.*

When we write $r:f\to g:\mathcal{A}\to\mathcal{B}$ we mean $r\in\mathcal{F}^\lambda\operatorname{Coder}(Ts\mathcal{A},\hat{T}s\mathcal{B})(f,g)^d$ for some $d\in\mathbb{Z}$ and $\lambda\in\mathbb{L}$. Suppose that $\tilde{h}:\hat{T}s\mathcal{B}\to\hat{T}s\mathcal{C}$ is a cofunctor. Then for r as above there

is a coderivation $r\tilde{h} \in \mathcal{F}^\lambda \operatorname{Coder}(Ts\mathcal{A}, \hat{T}s\mathcal{C})(f\tilde{h}, g\tilde{h})^d$, whose components are found as

$$(r\tilde{h})_l = \sum_{i_1+\dots+i_k+t+j_1+\dots+j_m=l}^{k,m\geqslant 0} \Big(T^l s\mathcal{A} \xrightarrow{f_{i_1}\otimes\dots\otimes f_{i_k}\otimes r_t\otimes g_{j_1}\otimes\dots\otimes g_{j_m}} (s\hat{\mathcal{B}})^{\otimes(k+1+m)} \xrightarrow{\ddot{\imath}} (s\hat{\mathcal{B}})^{\widehat{\otimes}(k+1+m)} \xrightarrow{\widehat{\ddot{\imath}^{\otimes k+1+m}}^{-1}} (s\mathcal{B})^{\widehat{\otimes}(k+1+m)} \xrightarrow{\tilde{h}_{k+1+m}} \hat{\mathfrak{c}} \Big),$$

due to Proposition 2.32, Remark 2.27 similarly to (2.13).

Suppose now that besides $\tilde{r} \in \mathcal{F}^\lambda \operatorname{Coder}(\hat{T}s\mathcal{A}, \hat{T}s\mathcal{B})(\tilde{f}, \tilde{g})^d$ we have a cofunctor $e : Ts\mathcal{C} \to \hat{T}s\mathcal{A}$. Then we have also a coderivation $e\tilde{r} : e\tilde{f} \to e\tilde{g} : \mathcal{C} \to \mathcal{B}$, $e\tilde{r} \in \mathcal{F}^\lambda \operatorname{Coder}(Ts\mathcal{C}, \hat{T}s\mathcal{B})(e\tilde{f}, e\tilde{g})^d$, whose components are given by

$$(e\tilde{r})_l = \sum_{i_1+\dots+i_k=l}^{k\geqslant 0} \big(T^l s\mathcal{C} \xrightarrow{e_{i_1}\otimes\dots\otimes e_{i_k}} (s\hat{\mathcal{A}})^{\otimes k} \xrightarrow{\ddot{\imath}} (s\hat{\mathcal{A}})^{\widehat{\otimes}k} \xrightarrow{\widehat{\ddot{\imath}^{\otimes k}}^{-1}} (s\mathcal{A})^{\widehat{\otimes}k} \xrightarrow{\tilde{r}_k} s\hat{\mathcal{B}} \big)$$

due to Theorem 2.23, Remark 2.27 similarly to (2.13).

2.37. Evaluation. Let $\mathfrak{A}$ be a completed conilpotent cocategory and let $\mathfrak{b}$ be a $\mathcal{V}$-quiver. Define the evaluation cofunctor $\operatorname{ev} : \mathfrak{A} \boxtimes T\operatorname{Coder}(\mathfrak{A}, \widehat{T\mathfrak{b}}) \to \widehat{T\mathfrak{b}}$ on objects as $\operatorname{ev}(A \boxtimes f) = fA$, and on morphisms as follows. Let $f^0, f^1, \dots, f^n : \mathfrak{A} \to \widehat{T\mathfrak{b}}$ be cofunctors, and let $r^1, \dots, r^n$ be coderivations of certain degrees and of some level as in $f^0 \xrightarrow{r^1} f^1 \xrightarrow{r^2} \dots f^{n-1} \xrightarrow{r^n} f^n : \mathfrak{A} \to \widehat{T\mathfrak{b}}$, $n \geqslant 0$. Then $c = r^1 \otimes \dots \otimes r^n \in T^n \operatorname{Coder}(\mathfrak{A}, \widehat{T\mathfrak{b}})(f^0, f^n)$. Define

$$[a \boxtimes (r^1 \otimes \dots \otimes r^n)] \operatorname{ev} = (a\Delta^{(2n+1)})(f^0 \hat{\otimes} \check{r}^1 \hat{\otimes} f^1 \hat{\otimes} \check{r}^2 \hat{\otimes} \dots \hat{\otimes} f^{n-1} \hat{\otimes} \check{r}^n \hat{\otimes} f^n) \mu^{(2n+1)}_{\widehat{T\mathfrak{b}}}.$$

The right hand side belongs to $(\widehat{T\mathfrak{b}})^{\hat{\otimes}(2n+1)} \mu^{(2n+1)}_{\widehat{T\mathfrak{b}}}$ and is mapped by multiplication $\mu^{(2n+1)}_{\widehat{T\mathfrak{b}}}$ from (A.4) into $\widehat{T\mathfrak{b}}$. So defined ev is a cofunctor. Indeed, $\eta \cdot \operatorname{ev} - \mathbb{1} \operatorname{ev} \cdot \eta$ applied to $1_A \boxtimes 1_f$, $A \in \operatorname{Ob} \mathfrak{A}$, $f \in \operatorname{Ob} \operatorname{Coder}(\mathfrak{A}, \widehat{T\mathfrak{b}}) = \operatorname{cncCat}(\mathfrak{A}, \widehat{T\mathfrak{b}})$, gives a tensor convergent expression in $\widehat{T\mathfrak{b}}$

$$[\eta(1_A) \boxtimes \operatorname{in}_0(1_f)] \operatorname{ev} - (1_{fA})\eta = f[\eta(1_A)] - \eta(1_{fA}),$$

since f is a cofunctor.

The following statement generalizes Proposition 3.4 of [Lyubashenko, 2003].

2.38. Theorem. *For* $\mathfrak{a} \in \operatorname{ncCat}$, $\mathfrak{b}, \mathfrak{c}^1, \dots \mathfrak{c}^q \in \Lambda\text{-mod}_{\mathbb{L}}\text{-}\mathsf{Quiv}$ *with notation* $\mathfrak{c} = T\mathfrak{c}^1 \boxtimes \dots \boxtimes T\mathfrak{c}^q$ *the map*

$$\operatorname{ncCat}(\mathfrak{c}, T\operatorname{Coder}(\mathfrak{a}, \widehat{T\mathfrak{b}})) \longrightarrow \operatorname{cncCat}(\mathfrak{a} \boxtimes \mathfrak{c}, \widehat{T\mathfrak{b}}),$$
$$\psi \longmapsto \big(\mathfrak{a} \boxtimes \mathfrak{c} \xrightarrow{\mathfrak{a}\boxtimes\psi} \mathfrak{a} \boxtimes T\operatorname{Coder}(\mathfrak{a}, \widehat{T\mathfrak{b}}) \xrightarrow{\operatorname{ev}} \widehat{T\mathfrak{b}} \big)$$

is a bijection.

PROOF. An augmentation preserving cofunctor $\psi : \mathfrak{c} \to T\,\mathrm{Coder}(\hat{\mathfrak{a}}, \widehat{T\mathfrak{b}}) \cong T\,\mathrm{Coder}(\mathfrak{a}, \widehat{T\mathfrak{b}})$ (see Corollary 2.36) is described by an arbitrary quiver map $\check{\psi} = \psi{\cdot}\mathrm{pr}_1 : \mathfrak{c} \to \mathrm{Coder}(\mathfrak{a}, \widehat{T\mathfrak{b}}) \in \Lambda\text{-mod}_{\mathbb{L}}\text{-}\mathsf{Quiv}$ such that $\eta \cdot \check{\psi} = 0$ by Proposition 1.9. Let $\phi : \mathfrak{a} \boxtimes \mathfrak{c} \to \widehat{T\mathfrak{b}}$ be a cofunctor. It equals the cofunctor $(\mathfrak{a} \boxtimes \psi) \cdot \mathrm{ev} : \mathfrak{a} \boxtimes \mathfrak{c} \to \widehat{T\mathfrak{b}}$ if the equation

$$\sum_{k\geqslant 0} (a \boxtimes c\Delta^{(k)}\check{\psi}^{\otimes k})\,\mathrm{ev} = (a \boxtimes c)\phi, \qquad a \in \mathfrak{a}^\bullet,\ c \in \mathfrak{c}^\bullet,$$

holds (by Theorem 2.23(i) and Proposition 2.26). It suffices to consider two cases. In the first one $c = \eta(1_g)$ for some $g \in \mathrm{Ob}\,\mathfrak{c}$. Then the equation takes the form $(a)(g\psi) = (a \boxtimes c)\phi$ which defines the cofunctor $g\psi \in \mathrm{cncCat}(\mathfrak{a}, \widehat{T\mathfrak{b}})$ in the left hand side.

In the second case $c \in \mathcal{F}^l\bar{\mathfrak{c}}^d$ the equation takes the form

$$(a)(c)\check{\psi} + \sum_{k\geqslant 2} (a \boxtimes c\Delta^{(k)}\check{\psi}^{\otimes k})\,\mathrm{ev} = (a \boxtimes c)\phi, \qquad a \in \mathfrak{a}^\bullet,\ c \in \bar{\mathfrak{c}}^\bullet.$$

Since $\eta \cdot \check{\psi} = 0$ the comultiplication Δ can be replaced with $\bar{\Delta}$. The structure of $\mathfrak{c} = T\mathfrak{c}^1 \boxtimes \cdots \boxtimes T\mathfrak{c}^q$ is such that the component $\psi_{i_1,\dots,i_q}$ in the left hand side of

$$(a)(c)\check{\psi} = (a \boxtimes c)\phi - \sum_{k\geqslant 2} (a \boxtimes c\bar{\Delta}^{(k)}\check{\psi}^{\otimes k})\,\mathrm{ev}, \qquad a \in \mathfrak{a}^\bullet,\ c \in \bar{\mathfrak{c}}^\bullet, \tag{2.18}$$

is expressed via the components $\psi_{j_1,\dots,j_q}$ with smaller indices $(j_1, \dots, j_q)$ in the product poset $\mathbb{N}^q$. For $c \in \mathcal{F}^l\bar{\mathfrak{c}}(X, Y)^d$, $X = (X_1, \dots, X_q)$, $Y = (Y_1, \dots, Y_q)$, $X_i, Y_i \in \mathrm{Ob}\,\mathfrak{c}^i$, find $n \geqslant 0$ such that $c\bar{\Delta}^{(n+1)} = 0$. Equation (2.18) determines a unique collection of maps $c\check{\psi} \in \mathcal{F}^l\underline{\Lambda\text{-mod}_{\mathbb{L}}}\big(\mathfrak{a}(U, V), \widehat{T\mathfrak{b}}((U, X)\phi, (V, Y)\phi)\big)^d$. It remains to verify that it is a coderivation. We have to prove that

$$(a)(c\check{\psi})\Delta_{\mathfrak{b}} = (a)\Delta_{\mathfrak{a}}[(\_ \boxtimes X)\phi \otimes (\_)(c\check{\psi}) + (\_)(c\check{\psi}) \otimes (\_ \boxtimes Y)\phi].$$

The case $n = 0$ being obvious, assume that $n \geqslant 1$. The sum in (2.18) goes from $k = 2$ to n. Correspondingly,

$$(a)(c\check{\psi})\Delta = (a\Delta)[(\_ \boxtimes c_{(1)})\phi \otimes (\_ \boxtimes c_{(2)})\phi] - \sum_{k=2}^{n} [(a\Delta) \boxtimes (c_{\bar{1}}\check{\psi} \otimes \cdots \otimes c_{\bar{k}}\check{\psi})\Delta_{T\,\mathrm{Coder}}]\tau_{(23)}(\mathrm{ev} \otimes \mathrm{ev}).$$

Here according to Sweedler's notation $c_{(1)} \otimes c_{(2)} = c\Delta$. Similarly, $c_{\bar{1}} \otimes \cdots \otimes c_{\bar{k}} = c\bar{\Delta}^{(k)}$. Recall the middle four interchange $[(a \otimes b) \boxtimes (c \otimes d)]\tau_{(23)} = (-1)^{bc}(a \boxtimes c) \otimes (b \boxtimes d)$. The above expression has to be equal to

$$(a\Delta)\Big\{(\_ \boxtimes 1_X)\phi \otimes \big[(\_ \boxtimes c)\phi - \sum_{k=2}^{n} \big(\_ \boxtimes (c_{\bar{1}}\check{\psi} \otimes \cdots \otimes c_{\bar{k}}\check{\psi})\big)\,\mathrm{ev}\big]\Big\}$$
$$+ (a\Delta)\Big\{\big[(\_ \boxtimes c)\phi - \sum_{k=2}^{n} \big(\_ \boxtimes (c_{\bar{1}}\check{\psi} \otimes \cdots \otimes c_{\bar{k}}\check{\psi})\big)\,\mathrm{ev}\big] \otimes (\_ \boxtimes 1_Y)\phi\Big\}.$$

Canceling the above terms we come to identity to be checked

$$(a\Delta)[(\_\boxtimes c_{\bar{1}})\phi\otimes(\_\boxtimes c_{\bar{2}})\phi]=\sum_{k=2}^{n}\big[(a\Delta)\boxtimes(c_{\bar{1}}\check{\psi}\otimes\cdots\otimes c_{\bar{k}}\check{\psi})\bar{\Delta}_{T\,\mathrm{Coder}}\big]\tau_{(23)}(\mathrm{ev}\otimes\mathrm{ev}).\quad(2.19)$$

The right hand side equals

$$\begin{aligned}
&\sum_{k=2}^{n}\sum_{i=1}^{k-1}\Big\{(a\Delta)\boxtimes\big[(c_{\bar{1}}\check{\psi}\otimes\cdots\otimes c_{\bar{i}}\check{\psi})\bigotimes(c_{\overline{i+1}}\check{\psi}\otimes\cdots\otimes c_{\bar{k}}\check{\psi})\big]\Big\}\tau_{(23)}(\mathrm{ev}\otimes\mathrm{ev})\\
&=\sum_{k=2}^{n}\sum_{i=1}^{k-1}(a\Delta)\Big\{\big[_\boxtimes(c_{\bar{1}}\check{\psi}\otimes\cdots\otimes c_{\bar{i}}\check{\psi})\big]\mathrm{ev}\otimes\big[_\boxtimes(c_{\overline{i+1}}\check{\psi}\otimes\cdots\otimes c_{\bar{k}}\check{\psi})\big]\,\mathrm{ev}\Big\}\\
&=\sum_{i=1}^{n}\sum_{j=1}^{n}(a\Delta)\Big\{\big[_\boxtimes(c_{\bar{1}}\check{\psi}\otimes\cdots\otimes c_{\bar{i}}\check{\psi})\big]\mathrm{ev}\otimes\big[_\boxtimes(c_{\overline{i+1}}\check{\psi}\otimes\cdots\otimes c_{\overline{i+j}}\check{\psi})\big]\,\mathrm{ev}\Big\}\\
&=(a\Delta)[(_)(c_{\bar{1}}F)\otimes(_)(c_{\bar{2}}F)],
\end{aligned}$$

where

$$(a)(cF)=\sum_{i=1}^{n}\big[a\boxtimes(c_{\bar{1}}\check{\psi}\otimes\cdots\otimes c_{\bar{i}}\check{\psi})\big]\mathrm{ev}=(a)(c\check{\psi})+\sum_{i=2}^{n}\big[a\boxtimes(c_{\bar{1}}\check{\psi}\otimes\cdots\otimes c_{\bar{i}}\check{\psi})\big]\mathrm{ev}=(a\boxtimes c)\phi$$

due to (2.18). Hence the right hand side of (2.19) equals $(a\Delta)[(\_\boxtimes c_{\bar{1}})\phi\otimes(\_\boxtimes c_{\bar{2}})\phi]$, which is the left hand side of (2.19). ∎

Let $\mathfrak{a}$ be a conilpotent cocategory and let $\mathfrak{b}$, $\mathfrak{c}$ be quivers. Consider the cofunctor given by the upper right path in the diagram

$$\begin{array}{ccc}
\mathfrak{a}\boxtimes T\,\mathrm{Coder}(\mathfrak{a},\widehat{T\mathfrak{b}})\boxtimes T\,\mathrm{Coder}(\widehat{T\mathfrak{b}},\widehat{T\mathfrak{c}}) & \xrightarrow{\mathrm{ev}\boxtimes 1} & \widehat{T\mathfrak{b}}\boxtimes T\,\mathrm{Coder}(\widehat{T\mathfrak{b}},\widehat{T\mathfrak{c}})\\
\Big\downarrow{\scriptstyle 1\boxtimes M} & = & \Big\downarrow{\scriptstyle \mathrm{ev}}\\
\mathfrak{a}\boxtimes T\,\mathrm{Coder}(\mathfrak{a},\widehat{T\mathfrak{c}}) & \xrightarrow{\qquad\mathrm{ev}\qquad} & \widehat{T\mathfrak{c}}
\end{array}$$

By Theorem 2.38 there is a unique augmentation preserving cofunctor

$$M: T\,\mathrm{Coder}(\mathfrak{a},\widehat{T\mathfrak{b}})\boxtimes T\,\mathrm{Coder}(\widehat{T\mathfrak{b}},\widehat{T\mathfrak{c}})\to T\,\mathrm{Coder}(\mathfrak{a},\widehat{T\mathfrak{c}}).$$

Denote by $\mathbb{1}$ the unit object $\boxtimes^0$ of the monoidal category of cocategories, that is, $\mathrm{Ob}\,\mathbb{1}=\{*\}$, $\mathbb{1}(*,*)=\Lambda$. Denote by $\mathbf{r}:\mathfrak{a}\boxtimes\mathbb{1}\to\mathfrak{a}$ and $\mathbf{l}:\mathbb{1}\boxtimes\mathfrak{a}\to\mathfrak{a}$ the corresponding natural isomorphisms. By Theorem 2.38 there exists a unique augmentation preserving cofunctor $\eta_{\widehat{T\mathfrak{b}}}:\mathbb{1}\to T\,\mathrm{Coder}(\widehat{T\mathfrak{b}},\widehat{T\mathfrak{b}})$, such that

$$\mathbf{r}=\big(\widehat{T\mathfrak{b}}\boxtimes\mathbb{1}\xrightarrow{1\boxtimes\eta_{\widehat{T\mathfrak{b}}}}\widehat{T\mathfrak{b}}\boxtimes T\,\mathrm{Coder}(\widehat{T\mathfrak{b}},\widehat{T\mathfrak{b}})\xrightarrow{\mathrm{ev}}\widehat{T\mathfrak{b}}\big).$$

Namely, the object $*\in\mathrm{Ob}\,\mathbb{1}$ goes to the identity homomorphism $\mathrm{id}_{\widehat{T\mathfrak{b}}}:\widehat{T\mathfrak{b}}\to\widehat{T\mathfrak{b}}$.

The following statement follows from Theorem 2.38.

2.39. Proposition. *The multiplication M is associative and η is its two-sided unit:*

$$\begin{array}{ccc} T\mathrm{Coder}(\mathfrak{a},\widehat{T\mathfrak{b}})\boxtimes T\mathrm{Coder}(\widehat{T\mathfrak{b}},\widehat{T\mathfrak{c}})\boxtimes T\mathrm{Coder}(\widehat{T\mathfrak{c}},\widehat{T\mathfrak{d}}) & \xrightarrow{M\boxtimes 1} & T\mathrm{Coder}(\mathfrak{a},\widehat{T\mathfrak{c}})\boxtimes T\mathrm{Coder}(\widehat{T\mathfrak{c}},\widehat{T\mathfrak{d}}) \\ \downarrow{\scriptstyle 1\boxtimes M} & & \downarrow{\scriptstyle M} \\ T\mathrm{Coder}(\mathfrak{a},\widehat{T\mathfrak{b}})\boxtimes T\mathrm{Coder}(\widehat{T\mathfrak{b}},\widehat{T\mathfrak{d}}) & \xrightarrow{\qquad M \qquad} & T\mathrm{Coder}(\mathfrak{a},\widehat{T\mathfrak{d}}) \end{array}$$

The multiplication M is computed explicitly in [Lyubashenko, 2003, §4], see, in particular, Examples 4.2 there.

3. Filtered A_∞-categories

For a filtered graded quiver $\mathcal{A}$ denote by $s\mathcal{A} = \mathcal{A}[1]$ the same quiver with the shifted grading, $\mathcal{A}[1]^n = \mathcal{A}^{n+1}$. The shift commutes with the completion. By s we denote also the "identity" map $s : \mathcal{A} \to \mathcal{A}[1]$, $\mathcal{A}^n \ni x \mapsto x \in \mathcal{A}[1]^{n-1}$, of degree -1.

3.1. Definition. *A filtered A_∞-category $\mathcal{A}$ is an $\mathbb{L}$-filtered $\mathbb{Z}$-graded quiver $\mathcal{A}$, equipped with a coderivation $b : \mathrm{Id} \to \mathrm{Id} : \hat{T}s\mathcal{A} \to \hat{T}s\mathcal{A}$ of degree 1 and of level 0, such that*[1] *the collection $b : \hat{T}s\mathcal{A}(X,Y) \to \hat{T}s\mathcal{A}(X,Y)$ satisfies $b^2 = 0$. Another name – curved A_∞-category. De Deken and Lowen use the name of filtered cA_∞-category [De Deken, Lowen, 2018].*

The codifferential b is determined in a unique fashion by the collection of morphisms $\check{b} = b \cdot \widehat{\mathrm{pr}_1} \in \mathcal{F}^0\underline{\Lambda\text{-mod}_{\mathbb{L}}}(\hat{T}s\mathcal{A}(X,Y), s\hat{\mathcal{A}}(X,Y))^1$, $X, Y \in \mathrm{Ob}\,\mathcal{A}$, equivalently, by the collection of morphisms $\check{b} = b \cdot \widehat{\mathrm{pr}_1} \in \mathcal{F}^0\underline{\Lambda\text{-mod}_{\mathbb{L}}}(Ts\mathcal{A}(X,Y), s\hat{\mathcal{A}}(X,Y))^1$, $X, Y \in \mathrm{Ob}\,\mathcal{A}$, due to Corollary 2.33, equivalently, by the components $b_n \in \mathcal{F}^0\underline{\Lambda\text{-mod}_{\mathbb{L}}}(T^n s\mathcal{A}(X,Y), s\hat{\mathcal{A}}(X,Y))^1$, $X, Y \in \mathrm{Ob}\,\mathcal{A}$, $n \geqslant 0$. The codifferential b is recovered from its components $b_j : T^j s\mathcal{A} \to s\hat{\mathcal{A}}$ due to Propositions 2.32, 2.35:

$$b = \sum_{i+j+k=n} (\ddot{\imath}1^{\hat{\otimes} i})\hat{\otimes} b_j \hat{\otimes} (\ddot{\imath}1^{\hat{\otimes} k}) : T^n s\mathcal{A} \to (T^{\leqslant n+1} s\mathcal{A})\widehat{\ }.$$

The square b^2 is a (1,1)-coderivation of level 0 and of degree 2:

$$b^2\Delta = b\Delta(1\hat{\otimes} b + b\hat{\otimes} 1) = \Delta(1\hat{\otimes} b + b\hat{\otimes} 1)^2 = \Delta(1\hat{\otimes} b^2 + b^2\hat{\otimes} 1) : Ts\mathcal{A} \to \hat{T}s\mathcal{A}.$$

Thus, the equation $b^2 = 0$ is equivalent to the system $(b^2)_n = 0$, $n \geqslant 0$. The components of b^2 can be found via Remark 2.27 by insertion of id between b and b. Therefore, the equation $b^2 = 0$ can be written as

$$\sum_{i+j+k=n} [(\ddot{\imath}1^{\hat{\otimes} i})\hat{\otimes} b_j \hat{\otimes} (\ddot{\imath}1^{\hat{\otimes} k})] b_{i+1+k} = 0 : T^n s\mathcal{A} \to s\hat{\mathcal{A}}.$$

[1] I am grateful to Kaoru Ono for explaining the reasons why the differential preserves the grading in Fukaya categories.

Let $\mathcal{A}$, $\mathcal{B}$ be filtered A_∞-categories, let $f^0, f^1, \dots, f^n : Ts\mathcal{A} \to \hat{T}s\mathcal{B}$ be cofunctors (see Definition 2.25), and let $r^1, \dots, r^n$ be coderivations of certain degrees and of some level as in $f^0 \xrightarrow{r^1} f^1 \xrightarrow{r^2} \dots f^{n-1} \xrightarrow{r^n} f^n : Ts\mathcal{A} \to \hat{T}s\mathcal{B}$, $n \geqslant 0$ (see Definition 2.34). Then $r^1 \otimes \dots \otimes r^n \in T^n \operatorname{Coder}(Ts\mathcal{A}, \hat{T}s\mathcal{B})(f^0, f^n)$. Let $a \in (T^\bullet s\mathcal{A})^\bullet$.

3.2. Proposition. *In the above assumptions there is a unique (1,1)-coderivation of degree 1 and level 0 $B : T \operatorname{Coder}(Ts\mathcal{A}, \hat{T}s\mathcal{B}) \to T \operatorname{Coder}(Ts\mathcal{A}, \hat{T}s\mathcal{B})$, such that*

$$[a \boxtimes (r^1 \otimes \dots \otimes r^n)] \operatorname{ev} b = [a \boxtimes (r^1 \otimes \dots \otimes r^n) B] \operatorname{ev} + (-)^{r^1 + \dots + r^n} [ab \boxtimes (r^1 \otimes \dots \otimes r^n)] \operatorname{ev} \quad (3.1)$$

for all $a \in Ts\mathcal{A}$, $n \geqslant 0$, $r^1 \otimes \dots \otimes r^n \in T^n \operatorname{Coder}(Ts\mathcal{A}, \hat{T}s\mathcal{B})(f^0, f^n)$. It satisfies $B^2 = 0$, thus, it gives an A_∞-structure to $s^{-1} \operatorname{Coder}(Ts\mathcal{A}, \hat{T}s\mathcal{B}) \cong s^{-1} \operatorname{Coder}(\hat{T}s\mathcal{A}, \hat{T}s\mathcal{B})$.

Proof. B is determined by its components $B_j : T^j \operatorname{Coder}(Ts\mathcal{A}, \hat{T}s\mathcal{B}) \to \operatorname{Coder}(Ts\mathcal{A}, \hat{T}s\mathcal{B})$ of degree 1 and level 0 due to Proposition 1.12:

$$B = \sum_{i+j+k=n} 1^{\otimes i} \otimes B_j \otimes 1^{\otimes k} : T^n \operatorname{Coder}(Ts\mathcal{A}, \hat{T}s\mathcal{B}) \to T^{\leqslant n+1} \operatorname{Coder}(Ts\mathcal{A}, \hat{T}s\mathcal{B}).$$

For $n = 0$ the equation reads $f^0 b = 1_{f^0} B + b f^0$, where $1_{f^0} = 1 \in T^0 \operatorname{Coder}(Ts\mathcal{A}, \hat{T}s\mathcal{B})(f^0, f^0) = \Lambda$. Hence, since both $f^0 b$ and $b f^0$ are (f^0, f^0)-coderivations, B_0 is found in a unique way as

$$1_{f^0} B_0 = f^0 b - b f^0 \in \operatorname{Coder}(Ts\mathcal{A}, \hat{T}s\mathcal{B})(f^0, f^0). \quad (3.2)$$

Assume that the coderivation components B_j for $j < n$ are already found so that (3.1) is satisfied up to $n - 1$ arguments. Let us determine a Λ-linear map $(r^1 \otimes \dots \otimes r^n) B_n : Ts\mathcal{A} \to \hat{T}s\mathcal{B}$ from (3.1) rewritten in the form

$$\begin{aligned} a.(r^1 \otimes \dots \otimes r^n) B_n = {} & [a \boxtimes (r^1 \otimes \dots \otimes r^n)] \operatorname{ev} b - (-)^{r^1 + \dots + r^n} [ab \boxtimes (r^1 \otimes \dots \otimes r^n)] \operatorname{ev} \\ & - \sum_{q+j+t=n}^{j<n} [a \boxtimes \{(r^1 \otimes \dots \otimes r^n)(1^{\otimes q} \otimes B_j \otimes 1^{\otimes t})\}] \operatorname{ev}. \end{aligned}$$

Let us show that $(r^1 \otimes \dots \otimes r^n) B_n$ is a (f^0, f^n)-coderivation. Indeed,

$$(r^1 \otimes \dots \otimes r^n) B_n \Delta = \Delta [f^0 \hat{\otimes} (r^1 \otimes \dots \otimes r^n) B_n + (r^1 \otimes \dots \otimes r^n) B_n \hat{\otimes} f^n] \quad (3.3)$$

due to computation

$$
\begin{aligned}
&a.(r^1 \otimes \cdots \otimes r^n)B_n\Delta = [a \boxtimes (r^1 \otimes \cdots \otimes r^n)]\Delta(\mathrm{ev}\,\hat{\otimes}\,\mathrm{ev})(1\hat{\otimes}b + b\hat{\otimes}1)\\
&-(-)^{r^1+\cdots+r^n}[ab \boxtimes (r^1 \otimes \cdots \otimes r^n)]\Delta(\mathrm{ev}\,\hat{\otimes}\,\mathrm{ev})\\
&-\sum_{q+j+t=n}^{j<n}[a \boxtimes \{(r^1 \otimes \cdots \otimes r^n)(1^{\otimes q} \otimes B_j \otimes 1^{\otimes t})\}]\Delta(\mathrm{ev}\,\hat{\otimes}\,\mathrm{ev}).\\
&=\sum_{k+l=n}(a\Delta)\{[\text{-} \boxtimes (r^1 \otimes \cdots \otimes r^k)]\,\mathrm{ev}\,\hat{\otimes}[\text{-} \boxtimes (r^{k+1} \otimes \cdots \otimes r^n)]\,\mathrm{ev}\}(1\hat{\otimes}b + b\hat{\otimes}1)\\
&-(-)^{r^1+\cdots+r^n}\sum_{k+l=n}(a\Delta)(1 \otimes b + b \otimes 1)\{[\text{-} \boxtimes (r^1 \otimes \cdots \otimes r^k)]\,\mathrm{ev}\,\hat{\otimes}[\text{-} \boxtimes (r^{k+1} \otimes \cdots \otimes r^n)]\,\mathrm{ev}\}\\
&-\sum_{q+j+t=n}^{j<n}(a\Delta)\sum_{k+v=q}[\text{-} \boxtimes (r^1 \otimes \cdots \otimes r^k)]\,\mathrm{ev}\,\hat{\otimes}[\text{-} \boxtimes \{(r^{k+1} \otimes \cdots \otimes r^n)(1^{\otimes v} \otimes B_j \otimes 1^{\otimes t})\}]\,\mathrm{ev}\\
&-\sum_{k=0}^{n}(-)^{r^{k+1}+\cdots+r^n}(a\Delta)\sum_{q+j+w=k}^{j<n}[\text{-} \boxtimes \{(r^1 \otimes \ldots \otimes r^k)(1^{\otimes q} \otimes B_j \otimes 1^{\otimes w})\}]\mathrm{ev}\hat{\otimes}[\text{-} \boxtimes (r^{k+1} \otimes \ldots \otimes r^n)]\mathrm{ev}\\
&=(a\Delta)\Big\langle\sum_{k+l=n}[\text{-} \boxtimes (r^1 \otimes \cdots \otimes r^k)]\,\mathrm{ev}\,\hat{\otimes}[\text{-} \boxtimes (r^{k+1} \otimes \cdots \otimes r^n)]\,\mathrm{ev}\,b\\
&-\sum_{k+l=n}(-)^{r^{k+1}+\cdots+r^n}[\text{-} \boxtimes (r^1 \otimes \cdots \otimes r^k)]\,\mathrm{ev}\,\hat{\otimes}[\text{-}b \boxtimes (r^{k+1} \otimes \cdots \otimes r^n)]\,\mathrm{ev}\\
&-\sum_{k+v+j+t=n}^{j<n}[\text{-} \boxtimes (r^1 \otimes \cdots \otimes r^k)]\,\mathrm{ev}\,\hat{\otimes}[\text{-} \boxtimes \{(r^{k+1} \otimes \cdots \otimes r^n)(1^{\otimes v} \otimes B_j \otimes 1^{\otimes t})\}]\,\mathrm{ev}\\
&+\sum_{k+l=n}(-)^{r^{k+1}+\cdots+r^n}[\text{-} \boxtimes (r^1 \otimes \cdots \otimes r^k)]\,\mathrm{ev}\,b\hat{\otimes}[\text{-} \boxtimes (r^{k+1} \otimes \cdots \otimes r^n)]\,\mathrm{ev}\\
&-\sum_{k+l=n}(-)^{r^1+\cdots+r^n}[\text{-}b \boxtimes (r^1 \otimes \cdots \otimes r^k)]\,\mathrm{ev}\,\hat{\otimes}[\text{-} \boxtimes (r^{k+1} \otimes \cdots \otimes r^n)]\,\mathrm{ev}\\
&-\sum_{k=0}^{n}(-)^{r^{k+1}+\cdots+r^n}\sum_{q+j+w=k}^{j<n}[\text{-} \boxtimes \{(r^1 \otimes \ldots \otimes r^k)(1^{\otimes q} \otimes B_j \otimes 1^{\otimes w})\}]\,\mathrm{ev}\,\hat{\otimes}[\text{-} \boxtimes (r^{k+1} \otimes \ldots \otimes r^n)]\,\mathrm{ev}\Big\rangle.
\end{aligned}
$$

The sum of the first three expressions in angle brackets equals its restriction to $k = 0$:

$$
\begin{aligned}
&f^0\hat{\otimes}[\text{-} \boxtimes (r^1 \otimes \cdots \otimes r^n)]\,\mathrm{ev}\,b - (-)^{r^1+\cdots+r^n}f^0\hat{\otimes}[\text{-}b \boxtimes (r^1 \otimes \cdots \otimes r^n)]\,\mathrm{ev}\\
&\qquad-\sum_{v+j+t=n}^{j<n}f^0\hat{\otimes}[\text{-} \boxtimes \{(r^1 \otimes \cdots \otimes r^n)(1^{\otimes v} \otimes B_j \otimes 1^{\otimes t})\}]\,\mathrm{ev} = f^0\hat{\otimes}(r^1 \otimes \cdots \otimes r^n)B_n.
\end{aligned}
$$

The sum of the last three expressions in angle brackets equals its restriction to $k = n$:

$$
[\text{-} \boxtimes (r^1 \otimes \cdots \otimes r^n)]\,\mathrm{ev}\,b\hat{\otimes}f^n - (-)^{r^1+\cdots+r^n}[\text{-}b \boxtimes (r^1 \otimes \cdots \otimes r^n)]\,\mathrm{ev}\,\hat{\otimes}f^n
$$

$$-\sum_{q+j+w=n}^{j<n}[\text{-}\boxtimes\{(r^1\otimes\cdots\otimes r^n)(1^{\otimes q}\otimes B_j\otimes 1^{\otimes w})\}]\operatorname{ev}\hat{\otimes}f^n=(r^1\otimes\cdots\otimes r^n)B_n\hat{\otimes}f^n.$$

This proves (3.3).

Notice that B^2 is a (1,1)-coderivation of level 0 and of degree 2:

$$B^2\Delta=B\Delta(1\otimes B+B\otimes 1)=\Delta(1\otimes B+B\otimes 1)^2=\Delta(1\otimes B^2+B^2\otimes 1).$$

Since $b^2=0$, we have for $a\in(T^\bullet s\mathcal{A})^\bullet$, $n\geqslant 0$, $r_i\in\operatorname{Coder}(Ts\mathcal{A},\hat{T}s\mathcal{B})$,

$$\begin{aligned}&[a\boxtimes(r^1\otimes\cdots\otimes r^n)B^2]\operatorname{ev}=[a\boxtimes(r^1\otimes\cdots\otimes r^n)B]\operatorname{ev}b-(-)^{r^1+\cdots+r^n+1}[ab\boxtimes(r^1\otimes\cdots\otimes r^n)B]\operatorname{ev}\\&=-(-)^{r^1+\cdots+r^n}[ab\boxtimes(r^1\otimes\cdots\otimes r^n)]\operatorname{ev}b-(-)^{r^1+\cdots+r^n+1}[ab\boxtimes(r^1\otimes\cdots\otimes r^n)]\operatorname{ev}b=0.\end{aligned}$$

Composing this equality with $\widehat{\operatorname{pr}_1}:\hat{T}s\mathcal{B}\to s\hat{\mathcal{B}}$ we get $0=[a\boxtimes(r^1\otimes\cdots\otimes r^n)B^2]\operatorname{ev}\widehat{\operatorname{pr}_1}$. Substituting into this expression the expansion of B^2 into components $B^2=\sum_{i+j+k=n}1^{\otimes i}\otimes(B^2)_j\otimes 1^{\otimes k}$, we find that all summands (except one), composed with ev, map $a\boxtimes(r^1\otimes\cdots\otimes r^n)$ into the ideal $\widehat{T^{\geqslant 2}s\mathcal{B}}$, and composed furthermore with $\widehat{\operatorname{pr}_1}$ vanish. The only surviving summand satisfies

$$0=[a\boxtimes(r^1\otimes\cdots\otimes r^n)(B^2)_n]\operatorname{ev}\widehat{\operatorname{pr}_1}=(a)[(r^1\otimes\cdots\otimes r^n)(B^2)_n]\widehat{\operatorname{pr}_1}.$$

By Corollary 2.33 the coderivation $(r^1\otimes\cdots\otimes r^n)(B^2)_n\in\operatorname{Coder}(Ts\mathcal{A},\hat{T}s\mathcal{B})(f^0,f^n)$ vanishes. Since this holds for all $n\geqslant 0$, the coderivation B^2 vanishes. ∎

3.3. Definition. *Let $\mathcal{A}$, $\mathcal{B}$ be filtered A_∞-categories. A cofunctor $f:Ts\mathcal{A}\to\hat{T}s\mathcal{B}$ is called a* filtered A_∞-functor *if $bf=fb$.*

Both sides of this equation are (f,f)-coderivations. In components:

$$\begin{aligned}(b\cdot f)_n&=\sum_{i+j+k=n}^{i,j,k\geqslant 0}(1^{\hat{\otimes}i}\hat{\otimes}b_j\hat{\otimes}1^{\hat{\otimes}k})\cdot f_{i+1+k}:T^ns\mathcal{A}\to s\hat{\mathcal{B}},\\(f\cdot b)_n&=\sum_{i_1+\cdots+i_k=n}^{k,i_j\geqslant 0}(f_{i_1}\hat{\otimes}\cdots\hat{\otimes}f_{i_k})\cdot b_k\quad:T^ns\mathcal{A}\to s\hat{\mathcal{B}}.\end{aligned}$$

Equality of these expressions for all $n\geqslant 0$ is equivalent to condition $bf=fb$ and, as we have seen, equivalent to $1_fB_0=0$.

Composing (3.1) with $\widehat{\operatorname{pr}_1}:\hat{T}s\mathcal{B}\to s\hat{\mathcal{B}}$ we find the components of coderivation $B:T\operatorname{Coder}(Ts\mathcal{A},\hat{T}s\mathcal{B})\to T\operatorname{Coder}(Ts\mathcal{A},\hat{T}s\mathcal{B})$. Recall that B_0 is given by (3.2), components of rB_1 for $r:f\to g:Ts\mathcal{A}\to\hat{T}s\mathcal{B}$ are found from

$$\begin{aligned}(a)(rB_1)\widehat{\operatorname{pr}_1}&=(a)[rb-(-)^rbr]\widehat{\operatorname{pr}_1}\\&=\sum_{i,k\geqslant 0}[a\Delta^{(i+1+k)}]\big[(\check{f}^{\otimes i}\otimes\check{r}\otimes\check{g}^{\otimes k})b_{i+1+k}-(-)^r(\operatorname{pr}_1^{\otimes i}\otimes\check{b}\otimes\operatorname{pr}_1^{\otimes k})r_{i+1+k}\big].\end{aligned}$$

Notice that, in general, $[r,b] \equiv rb - (-)^r br$ is not a coderivation, unless the source and the target of r are A_∞-functors (cf. Remark 3.4). In detail, denote by $(a)(rB_1)^\vee$ the coderivation value $(a)(rB_1)\widehat{\mathrm{pr}_1}$. Then by Remark 2.27

$$(rB_1)^\vee_0 = \sum_{i,k\geqslant 0} (f_0^{\otimes i} \otimes r_0 \otimes g_0^{\otimes k}) b_{i+1+k} - (-)^r b_0 r_1,$$

$$(rB_1)^\vee_1 = \sum_{i,k\geqslant 0} (f_0^{\otimes i} \otimes r_1 \otimes g_0^{\otimes k}) b_{i+1+k} + \sum_{m,n,k\geqslant 0} (f_0^{\otimes m} \otimes f_1 \otimes f_0^{\otimes n} \otimes r_0 \otimes g_0^{\otimes k}) b_{m+n+2+k}$$
$$+ \sum_{i,m,n\geqslant 0} (f_0^{\otimes i} \otimes r_0 \otimes g_0^{\otimes m} \otimes g_1 \otimes g_0^{\otimes n}) b_{i+2+m+n} - (-)^r [b_1 r_1 + (1\otimes b_0) r_2 + (b_0 \otimes 1) r_2],$$

etc. For $n \geqslant 2$ we have

$$(a)[(r^1 \otimes \cdots \otimes r^n) B_n] \widehat{\mathrm{pr}_1} = [a \boxtimes (r^1 \otimes \cdots \otimes r^n)] \operatorname{ev} b \, \widehat{\mathrm{pr}_1} =$$
$$\sum_{i^0,i^1,\ldots,i^n\geqslant 0} [a\Delta^{(i^0+\cdots+i^n+n)}] \big((\check f^0)^{\otimes i^0} \otimes \check r^1 \otimes (\check f^1)^{\otimes i^1} \otimes \check r^2 \otimes \cdots \otimes (\check f^{n-1})^{\otimes i^{n-1}} \otimes \check r^n \otimes (\check f^n)^{\otimes i^n} \big) b_{i^0+\cdots+i^n+n}.$$

3.4. Remark. Let $f, g : Ts\mathcal{A} \to \hat{T}s\mathcal{B}$ be filtered A_∞-functors and let r be an (f,g)-coderivation of degree d and of level l. Then $[r,b] = rb - (-)^r br$ is an (f,g)-coderivation of degree $d+1$ and of level l, in particular, $rB_1 = [r,b]$. Indeed,

$$\begin{aligned}(rb - (-)^r br)\Delta &= r\Delta(1\hat\otimes b + b\hat\otimes 1) - (-)^r b\Delta(f\hat\otimes r + r\hat\otimes g)\\ &= \Delta[(f\hat\otimes r + r\hat\otimes g)(1\hat\otimes b + b\hat\otimes 1) - (-)^r (1\hat\otimes b + b\hat\otimes 1)(f\hat\otimes r + r\hat\otimes g)]\\ &= \Delta[f\hat\otimes(rb - (-)^r br) + (-)^r (fb - bf)\hat\otimes r + r\hat\otimes(gb - bg) + (rb - (-)^r br)\hat\otimes g].\end{aligned}$$

A. Reflective representable multicategories

Let $\mathcal{V}$ be a symmetric monoidal category, for instance, $\mathcal{V} = \mathsf{grAb}$. Let D be a lax representable plain/symmetric/braided $\mathcal{V}$-multicategory [Bespalov, Lyubashenko, Manzyuk, 2008, Definitions 3.7, 3.23], that is, for all families $(M_i)_{i\in I}$ of objects of D the $\mathcal{V}$-functors $\mathsf{D}((M_i)_{i\in I}; -) : \mathsf{D} \to \mathcal{V}$ are representable. By [Bespalov, Lyubashenko, Manzyuk, 2008, Theorem 3.24] the $\mathcal{V}$-multicategory D is isomorphic to $\mathcal{V}$-multicategory $\widehat{\mathcal{D}}$ for a lax plain/symmetric/braided monoidal $\mathcal{V}$-category $\mathcal{D} = (\mathcal{D}, \otimes^I, \lambda^f, \rho^L)$. The $\mathcal{V}$-multicategory $\widehat{\mathcal{D}}$ has $\widehat{\mathcal{D}}((M_i)_{i\in I}; N) = \mathcal{D}(\otimes^I(M_i), N)$ (see [Bespalov, Lyubashenko, Manzyuk, 2008, Proposition 3.22] for details). We may and we will take for $\mathcal{D}$ the category D, that is, $\operatorname{Ob}\mathcal{D} = \operatorname{Ob}\mathsf{D}$, $\mathcal{D}(M,N) = \mathsf{D}(M;N)$. Denote by $\tilde{\mathsf{D}}$ the plain/symmetric/braided multicategory with $\operatorname{Ob}\tilde{\mathsf{D}} = \operatorname{Ob}\mathsf{D}$, $\tilde{\mathsf{D}}((M_i)_{i\in I}; N) = \mathcal{V}(\mathbb{1}_\mathcal{V}, \mathsf{D}((M_i)_{i\in I}; N))$. For instance, when $\mathcal{V} = \mathsf{grAb}$ we have $\tilde{\mathsf{D}} = \mathsf{D}^0$. Instead of morphism $f : \mathbb{1}_\mathcal{V} \to \mathsf{D}((M_i)_{i\in I}; N) \in \mathcal{V}$ we write $f : (M_i)_{i\in I} \to N \in \tilde{\mathsf{D}}$. The multicategory $\tilde{\mathsf{D}}$ is represented by the lax plain/symmetric/braided monoidal category $\tilde{\mathcal{D}}$ with $\operatorname{Ob}\tilde{\mathcal{D}} = \operatorname{Ob}\mathcal{D}$, $\tilde{\mathcal{D}}(M,N) = \mathcal{V}(\mathbb{1}_\mathcal{V}, \mathcal{D}(M,N)) = \tilde{\mathsf{D}}(M;N)$.

Assume that $\operatorname{Ob}\mathsf{D}$ contains a subset $\operatorname{Ob}\mathcal{C}$ such that the full subcategory $\mathcal{C} \subset \mathcal{D}$ is reflective. Recall that this is equivalent to giving a morphism $\ddot\imath_M : M \to \hat M \in \tilde{\mathcal{D}}$ for every

$M \in \operatorname{Ob}\mathcal{D}$, where $\hat{M} \in \operatorname{Ob}\mathcal{C}$ and for all $N \in \operatorname{Ob}\mathcal{C}$ the morphism $\mathcal{D}(\ddot{\imath}_M, N) : \mathcal{D}(\hat{M}, N) \to \mathcal{D}(M, N)$ is invertible. In other words, the inclusion $\mathcal{V}$-functor $\mathrm{in} : \mathcal{C} \hookrightarrow \mathcal{D}$ has a left adjoint $\widehat{\text{-}} : \mathcal{D} \to \mathcal{C}$. The unit of this adjunction is $\ddot{\imath} : \mathrm{Id}_{\mathcal{D}} \to \mathrm{in}\circ\widehat{\text{-}}$. Denote by C the full plain/symmetric/braided $\mathcal{V}$-submulticategory of D with $\operatorname{Ob}\mathsf{C} = \operatorname{Ob}\mathcal{C}$.

A.1. Proposition. *The $\mathcal{V}$-multicategory C is lax representable by a lax plain/symmetric/braided monoidal $\mathcal{V}$-category $\mathcal{C} = (\mathcal{C}, \hat{\otimes}^I, \hat{\lambda}^f, \hat{\rho}^L)$ with*

$$\hat{\otimes}^{i\in I} M_i = \widehat{\otimes^{i\in I} M_i}, \tag{A.1}$$

$$\hat{\lambda}^f = \big[\widehat{\otimes^{i\in I} M_i} \xrightarrow{\widehat{\lambda^f}} (\otimes^{j\in J} \otimes^{i\in f^{-1}j} M_i)\widehat{\ } \xrightarrow{\widehat{\otimes^{j\in J}\ddot{\imath}}} (\otimes^{j\in J}(\otimes^{i\in f^{-1}j} M_i)\widehat{\ })\widehat{\ }\big], \tag{A.2}$$

$$\hat{\rho}^L = \big(\hat{\otimes}^L M = \widehat{\otimes^L M} \xrightarrow{\widehat{\rho^L}} \hat{M} \xrightarrow{\ddot{\imath}_M^{-1}} M\big). \tag{A.3}$$

Proof. Without loss of generality we assume that $\mathsf{D} = \widehat{\mathcal{D}}$, that is, $\mathsf{D}((M_i)_{i\in I}; N) = \mathcal{D}(\otimes^{i\in I} M_i, N)$. In particular, $\tau : (M_i)_{i\in I} \to \otimes^{i\in I} M_i \in \tilde{\mathsf{D}}$ corresponds to $\mathrm{id}_{\otimes^{i\in I} M_i} \in \tilde{\mathcal{D}}$. Supposing that $N, M_i \in \operatorname{Ob}\mathsf{C}$ for $i \in I$ we get isomorphisms

$$\mathsf{C}(\hat{\tau}; N) = \big[\mathsf{C}(\widehat{\otimes^{i\in I} M_i}; N) \xrightarrow{\mathsf{D}(\ddot{\imath}; N)} \mathsf{D}(\otimes^{i\in I} M_i; N) \xrightarrow{\mathsf{D}(\tau; N)} \mathsf{C}((M_i)_{i\in I}; N)\big],$$

where

$$\hat{\tau} = \big[(M_i)_{i\in I} \xrightarrow{\tau} \otimes^{i\in I} M_i \xrightarrow{\ddot{\imath}} \widehat{\otimes^{i\in I} M_i}\big] \in \tilde{\mathsf{C}}.$$

Therefore, C is lax representable.

By the proof of Theorem 3.24 of [Bespalov, Lyubashenko, Manzyuk, 2008] the lax plain/symmetric/braided monoidal $\mathcal{V}$-category $(\mathcal{C}, \hat{\otimes}^I, \hat{\lambda}^f, \hat{\rho}^L)$ has structure elements given precisely by (A.1)–(A.3). For instance, an expression from [*ibid.*] is the top-right path in the following diagram

$$\begin{array}{ccccc}
\otimes^{i\in I} M_i & \xrightarrow{\lambda^f} & \otimes^{j\in J} \otimes^{i\in f^{-1}j} M_i & \xrightarrow{\otimes^{j\in J}\ddot{\imath}} & \otimes^{j\in J}(\otimes^{i\in f^{-1}j} M_i)\widehat{\ } \\
\downarrow{\scriptstyle \ddot{\imath}} & & \downarrow{\scriptstyle \ddot{\imath}} & & \downarrow{\scriptstyle \ddot{\imath}} \\
\widehat{\otimes^{i\in I} M_i} & \xrightarrow{\widehat{\lambda^f}} & (\otimes^{j\in J} \otimes^{i\in f^{-1}j} M_i)\widehat{\ } & \xrightarrow{\widehat{\otimes^{j\in J}\ddot{\imath}}} & (\otimes^{j\in J}(\otimes^{i\in f^{-1}j} M_i)\widehat{\ })\widehat{\ }
\end{array}$$

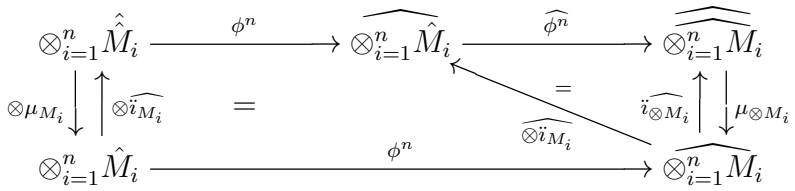
The same expression has to be equal to $\ddot{\imath} \cdot \hat{\lambda}^f$. Since the diagram commutes, $\hat{\lambda}^f$ is equal to the bottom composition $\widehat{\lambda^f} \cdot \widehat{\otimes^{j\in J}\ddot{\imath}}$. ∎

Since $\mathcal{C} \subset \mathcal{D}$ is a full reflective subcategory there is an idempotent monad $\hat{\text{-}} : \mathcal{D} \to \mathcal{D}$, $M \mapsto \hat{M}$, with the unit $\ddot{\imath} : \mathrm{Id}_{\mathcal{D}} \to \hat{\text{-}}$, $M \to \hat{M}$, and multiplication $\mu_M : \hat{\hat{M}} \to \hat{M}$ inverse to $\ddot{\imath}_{\hat{M}} = \hat{\ddot{\imath}}_M : \hat{M} \to \hat{\hat{M}}$ [Borceux, 1994, Corollary 4.2.4] (see enriched version at the end of Chapter 1 of [Kelly, 1982]).

Assume furthermore that $\hat{\text{-}}$ extends to a lax plain/symmetric/braided monoidal functor $(\hat{\text{-}}, \phi^n)$, $\phi^n : \otimes_{i=1}^n \hat{M}_i \to \widehat{\otimes_{i=1}^n M_i}$, and the unit $\ddot{\imath}$ satisfies condition (2.5):

$$\big(M_1 \otimes \cdots \otimes M_n \xrightarrow{\ddot{\imath}_1 \otimes \cdots \otimes \ddot{\imath}_n} \hat{M}_1 \otimes \cdots \otimes \hat{M}_n \xrightarrow{\phi^n} \widehat{M_1 \otimes \cdots \otimes M_n}\big) = \ddot{\imath}_{M_1 \otimes \cdots \otimes M_n}.$$

That is, $\ddot{\imath}$ is a monoidal transformation in the sense of [Bespalov, Lyubashenko, Manzyuk, 2008, Definition 2.7]. Therefore, $\mu : (\hat{\cdot}, \phi^n)^2 \to (\hat{\cdot}, \phi^n)$ is a monoidal transformation as well. Indeed, this follows from the commutative diagram



Summing up, $((\hat{\cdot}, \phi^n), \ddot{\imath}, \mu)$ is an idempotent lax plain/symmetric/braided monoidal monad.

Let $\{1, 2, \dots, n\} = S \sqcup P$. Given $M_i \in \operatorname{Ob} \mathcal{D}$, define

$$N_i = \begin{cases} M_i, & \text{if } i \in S, \\ \hat{M}_i, & \text{if } i \in P. \end{cases}$$

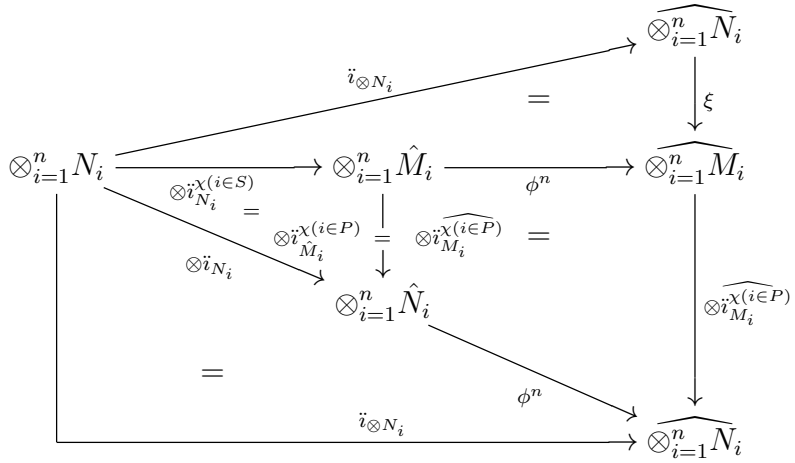
Let $\ddot{\imath}^0 = 1$, $\ddot{\imath}^1 = \ddot{\imath}$. Define

$$\chi(i \in P) = \begin{cases} 0, & \text{if } i \in S, \\ 1, & \text{if } i \in P \end{cases}$$

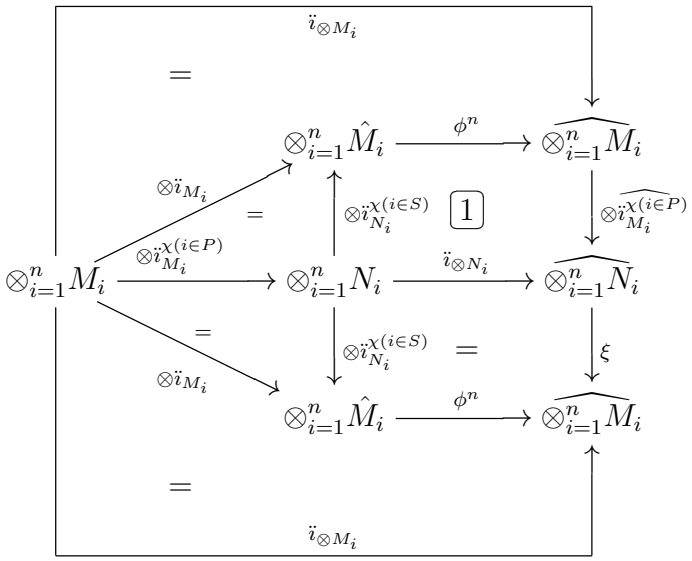
and $\chi(i \in S) = 1 - \chi(i \in P)$. Similarly to [De Deken, Lowen, 2018, Proposition 2.27] we prove

A.2. Proposition. *The morphism* $\widehat{\otimes \ddot{\imath}^{\chi(i\in P)}_{M_i}} : \widehat{\otimes^n_{i=1} M_i} \to \widehat{\otimes^n_{i=1} N_i}$ *is invertible.*

Proof. There is a unique morphism $\xi : \widehat{\otimes^n_{i=1} N_i} \to \widehat{\otimes^n_{i=1} M_i}$ which forces the diagram



to commute. The reflectivity implies that $\xi\cdot\widehat{\otimes\ddot{i}_{M_i}^{\chi(i\in P)}}=1$. Commutativity of the composite rectangle in this diagram implies commutativity of square $\boxed{1}$ in the following diagram



Commutativity of the above together with reflectivity implies that $\widehat{\otimes\ddot{i}_{M_i}^{\chi(i\in P)}}\cdot\xi=1$. Thus, ξ is inverse to $\widehat{\otimes\ddot{i}_{M_i}^{\chi(i\in P)}}$. ∎

The unit object of $(\mathcal{D},\otimes^n)$ is $\mathbb{1}=\otimes^0(*)$, therefore, the unit object of $(\mathcal{C},\hat{\otimes}^n)$ is $\hat{\mathbb{1}}=\hat{\otimes}^0(*)$.

A.3. Corollary. *When $\mathcal{D}$ is a plain/symmetric/braided monoidal category (all λ^f, ρ^L are invertible), so is $\mathcal{C}$.*

In fact, invertibility of $\widehat{\lambda^f}$ and of $\widehat{\otimes^{j\in J}\ddot{\imath}}$ implies invertibility of their composition $\hat{\lambda}^f$.

A.4. Algebras and coalgebras. Assume that the category $\mathcal{D}$ is monoidal (all λ^f, ρ^L are invertible). Hence, the same for $\mathcal{C}$. The category of algebras (monoids) in $\mathcal{D}$ (resp. $\mathcal{C}$) is denoted $\mathrm{Alg}_{\tilde{\mathcal{D}}}$ (resp. $\mathrm{Alg}_{\tilde{\mathcal{C}}}$).

A.5. Proposition. *The full and faithful functor* $\mathrm{in}:\mathrm{Alg}_{\tilde{\mathcal{C}}}\to\mathrm{Alg}_{\tilde{\mathcal{D}}}$, $(B,\mu_B:B\hat{\otimes}B\to B,\eta_B:\hat{\mathbb{1}}\to B)\mapsto\big(B,B\otimes B\xrightarrow{\ddot{\imath}}\widehat{B\otimes B}\xrightarrow{\mu_B}B,\mathbb{1}\xrightarrow{\ddot{\imath}}\hat{\mathbb{1}}\xrightarrow{\eta_B}B\big)$ *turns* $\mathrm{Alg}_{\tilde{\mathcal{C}}}$ *into a reflective subcategory of* $\mathrm{Alg}_{\tilde{\mathcal{D}}}$.

Proof. First of all, $\mathrm{in}\,B$ is an algebra in $\mathcal{D}$ (the proof is left to the reader). Secondly, any morphism $f:A\to B\in\mathrm{Alg}_{\tilde{\mathcal{C}}}$ induces $f:\mathrm{in}\,A\to\mathrm{in}\,B\in\mathrm{Alg}_{\tilde{\mathcal{D}}}$. Clearly, the functor in is faithful. One can show that it is full. This functor has a left adjoint, namely, the completion functor $\hat{\cdot}:\mathrm{Alg}_{\tilde{\mathcal{D}}}\to\mathrm{Alg}_{\tilde{\mathcal{C}}}$, $(A,\mu_A,\eta_A)\mapsto\big(\hat{A},\mu_{\hat{A}}=(\hat{A}\hat{\otimes}\hat{A}=\widehat{\hat{A}\otimes\hat{A}}\xrightarrow{\widehat{\ddot{\imath}\otimes\ddot{\imath}}^{-1}}\widehat{A\otimes A}\xrightarrow{\widehat{\mu_A}}\hat{A}),\eta_{\hat{A}}=\widehat{\eta_A}:\hat{\mathbb{1}}\to\hat{A}\big)$. The natural transformation $\ddot{\imath}:A\to\hat{A}=\big(\hat{A},\hat{A}\otimes\hat{A}\xrightarrow{\ddot{\imath}}\widehat{\hat{A}\otimes\hat{A}}\xrightarrow{\widehat{\ddot{\imath}\otimes\ddot{\imath}}^{-1}}\widehat{A\otimes A}\xrightarrow{\widehat{\mu_A}}\hat{A},\mathbb{1}\xrightarrow{\ddot{\imath}}\hat{\mathbb{1}}\xrightarrow{\widehat{\eta_A}}\hat{A}\big)$ is given precisely by $\ddot{\imath}:A\to\hat{A}$.

The proof of these statement is left to the reader. ∎

Let $\tilde{\mathcal{D}}$ contain arbitrary small coproducts. Then for any $M \in \operatorname{Ob}\mathcal{D}$ there is the tensor algebra $TM = \coprod_{n\geqslant 0} M^{\otimes n} \equiv \oplus_{n\geqslant 0} M^{\otimes n} \in \tilde{\mathcal{D}}$. The functor $T : \tilde{\mathcal{D}} \to \mathrm{Alg}_{\tilde{\mathcal{D}}}$ is left adjoint to the underlying functor $\mathcal{U} : \mathrm{Alg}_{\tilde{\mathcal{D}}} \to \tilde{\mathcal{D}}$. Then for $B \in \mathrm{Alg}_{\tilde{\mathcal{C}}}$, $X \in \mathcal{C}$ there are natural bijections

$$\mathrm{Alg}_{\tilde{\mathcal{C}}}(\widehat{TX}, B) \cong \mathrm{Alg}_{\tilde{\mathcal{D}}}(TX, \mathrm{in}\, B) \cong \tilde{\mathcal{D}}(X, \mathcal{U}\,\mathrm{in}\, B) = \tilde{\mathcal{C}}(X, \mathcal{U}B).$$

Hence, the functor $\tilde{\mathcal{C}} \to \mathrm{Alg}_{\tilde{\mathcal{C}}}$, $X \mapsto \widehat{TX}$ is left adjoint to $\mathcal{U} : \mathrm{Alg}_{\tilde{\mathcal{C}}} \to \tilde{\mathcal{C}}$. Multiplication in the algebra $\hat{A} = \widehat{TX}$ with $A = TX$

$$\mu^{(I)}_{\hat{A}} = \big[(\widehat{TX})^{\hat{\otimes} I} = \widehat{(\widehat{TX})^{\otimes I}} \xrightarrow{\widehat{i^{\otimes I}}^{-1}} \widehat{(TX)^{\otimes I}} \xrightarrow{\widehat{\mu^{(I)}_A}} \widehat{TX}\big] \tag{A.4}$$

is denoted also as $\hat{\otimes}$ (by abuse of notation).

A.5.1. Completion of coalgebras. The category of coalgebras (comonoids) in $\mathcal{D}$ (resp. $\mathcal{C}$) is denoted $\mathrm{Coalg}_{\tilde{\mathcal{D}}}$ (resp. $\mathrm{Coalg}_{\tilde{\mathcal{C}}}$). The completion functor extends to a functor $\hat{\cdot} : \mathrm{Coalg}_{\tilde{\mathcal{D}}} \to \mathrm{Coalg}_{\tilde{\mathcal{C}}}$, $(C, \Delta_C, \varepsilon_C) \mapsto \big(\hat{C}, \Delta_{\hat{C}} = (\hat{C} \xrightarrow{\widehat{\Delta_C}} \widehat{C \otimes C} \xrightarrow{\widehat{i\otimes i}} \widehat{\hat{C} \otimes \hat{C}} = \hat{C}\hat{\otimes}\hat{C}), \varepsilon_{\hat{C}} = \widehat{\varepsilon_C} : \hat{C} \to \hat{\mathbb{1}}\big)$. The proof is left to the reader. Notice that

$$\big(\hat{C} \xrightarrow{\widehat{\Delta^{(k)}_C}} \widehat{C^{\otimes k}} \xrightarrow{\widehat{i^{\otimes k}}} \widehat{\hat{C}^{\otimes k}} = \hat{C}^{\hat{\otimes} k}\big) = \Delta^{(k)}_{\hat{C}}.$$

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