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SPAN COMPOSITION USING FAKE PULLBACKS

Dedicated to Bob Rosebrugh, categorical communication catalyst

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ABSTRACT. The construction of a category of spans can be made in some categories $\mathscr{A}$ which do not have pullbacks in the traditional sense. The PROP for monoids is a good example of such an $\mathscr{A}$. The 2012 book concerning homological algebra by Marco Grandis gives the proof of associativity of relations in a Puppe-exact category based on a 1967 paper of M.Š. Calenko. The proof here is a restructuring of that proof in the spirit of the first sentence of this Abstract. We observe that these relations are spans of EM-spans and that EM-spans admit fake pullbacks so that spans of EM-spans compose. Our setting is more general than Puppe-exact categories. We mention the formalism of distributive laws which, in a generalized form, would cover our setting.

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Introduction

The construction of a category of spans can be made in some categories $\mathscr{A}$ not having pullbacks in the traditional sense, only having some form of *fake pullback*. The PROP for monoids is a good example of such an $\mathscr{A}$; it has a forgetful functor to the category of finite sets which takes fake pullbacks to genuine pullbacks.

As discussed in the book [16] by Marco Grandis, relations in a Puppe-exact category $\mathscr{C}$ are zig-zag diagrams of monomorphisms and epimorphisms, rather than the jointly monomorphic spans as for a regular category (see [7] for example). Associativity of these

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zig-zag relations was proved by M.Š. Calenko [6] over 50 years ago; also see [5] Appendix A.5, pages 140–142.

The present paper is a restructuring of the associativity proof in the spirit of fake pullbacks. The original category $\mathscr{C}$ does not even need to be pointed, but it should have a *suitable factorization system* $(\mathscr{E}, \mathscr{M})$. The fake pullbacks are constructed in what we call $\mathrm{Spn}(\mathscr{E}, \mathscr{M})$, not in $\mathscr{C}$ itself, and there is no forgetful functor turning them into genuine pullbacks. The relations are spans in $\mathrm{Spn}(\mathscr{E}, \mathscr{M})$. The main point in proving associativity of the span composition is that fake pullbacks stack properly.

Furthermore, we relate fake pullbacks to distributive laws in the sense of Jon Beck [2]. We are grateful to the referee for suggesting we include material on this.

1. Suitable factorization systems

Let $(\mathscr{E}, \mathscr{M})$ be a factorization system in the sense of Freyd-Kelly [13] on a category $\mathscr{C}$. That is, $\mathscr{E}$ and $\mathscr{M}$ are sets of morphisms of $\mathscr{C}$ which satisfy the conditions:

FS0. if $m \in \mathscr{M}$ and w is invertible then $mw \in \mathscr{M}$ while if $e \in \mathscr{E}$ and w is invertible then $we \in \mathscr{E}$;

FS1. if $mu = ve$ with $e \in \mathscr{E}$ and $m \in \mathscr{M}$ then there exists a unique w with $we = u$ and $mw = v$;

FS2. every morphism f factorizes $f = m \circ e$ with $e \in \mathscr{E}$ and $m \in \mathscr{M}$.

If we write $f : A \twoheadrightarrow B$, we mean $f \in \mathscr{E}$. If we write $f : A \rightarrowtail B$, we mean $f \in \mathscr{M}$. Another way to express FS1 is to ask, for all $X \xrightarrow{e} Y \in \mathscr{E}$ and $A \xrightarrow{m} B \in \mathscr{M}$, that the square (1.1) should be a pullback.

$$\begin{tikzcd} \mathscr{C}(Y,A) \arrow[r, "\mathscr{C}(e,A)"] \arrow[d, "\mathscr{C}(Y,m)"'] & \mathscr{C}(X,A) \arrow[d, "\mathscr{C}(X,m)"] \\ \mathscr{C}(Y,B) \arrow[r, "\mathscr{C}(e,B)"'] & \mathscr{C}(X,B) \end{tikzcd} \tag{1.1}$$

1.1. Remark.

(a) Actually Freyd-Kelly [13] included the assumption, which in isolation from the other conditions is clearly stronger than F0, that $\mathscr{E}$ and $\mathscr{M}$ are closed under composition (see Ehrbar-Wyler [12]).

(b) If we were dealing with a factorization system on a bicategory $\mathscr{C}$, we would ask (1.1) (with the associativity constraint providing a natural isomorphism in the square) to be a bipullback. Also, in FS2, we would only ask $f \cong m \circ e$; see Dupont-Vitale [11]. This is relevant to Proposition 2.4 and Section 5 below.

The factorization system $(\mathscr{E}, \mathscr{M})$ is *suitable* when it satisfies:

SFS1. pullbacks of arbitrary morphisms along members of $\mathscr{M}$ exist;

SFS2. pushouts of arbitrary morphisms along members of $\mathscr{E}$ exist;

SFS3. the pullback of an $\mathscr{E}$ along an $\mathscr{M}$ is in $\mathscr{E}$;

SFS4. the pushout of an $\mathscr{M}$ along an $\mathscr{E}$ is in $\mathscr{M}$;

SFS5. a commutative square of the form

$$\begin{tikzcd} D \arrow[r, tail, "n"] \arrow[d, two heads, "e"'] & B \arrow[d, two heads, "d"] \\ A \arrow[r, tail, "m"'] & C \end{tikzcd}$$

is a pullback if and only if it is a pushout.

1.2. Proposition.

(i) Spans of the form $X \twoheadleftarrow S \rightarrowtail Y$ *are jointly monomorphic.*

(ii) Cospans of the form $X \rightarrowtail C \twoheadleftarrow Y$ *are jointly epimorphic.*

Proof. A pullback of $X \rightarrowtail C \twoheadleftarrow Y$ exists by SFS1 and $X \rightarrowtail C \twoheadleftarrow Y$ is the pushout of the resultant span by SFS5. Pushout cospans are jointly epimorphic. This proves (ii), and (i) is dual. ■

1.3. Example. [Suitable factorization systems]

1. Take $\mathscr{C}$ to be the category Grp of groups, $\mathscr{E}$ to be the set of surjective morphisms and $\mathscr{M}$ to be the set of injective morphisms.

2. Take $\mathscr{C}$ to be any Puppe-exact category as studied by Grandis [16], $\mathscr{E}$ the epimorphisms and $\mathscr{M}$ the monomorphisms. This includes all abelian categories.

3. Take $\mathscr{C}$ to be the category of spans in the category $\mathrm{Set}_{\mathrm{inj}}$ of sets and injective functions. The objects are sets and the morphisms are isomorphism classes of spans with both legs injective functions. Take $\mathscr{E}$ to consist of the morphisms represented by spans i^* with right leg an identity and $\mathscr{M}$ represented by spans i_* with left leg an identity.

4. Take $\mathscr{C}$ to be any groupoid with $\mathscr{E} = \mathscr{M}$ containing all morphisms. This is a special case of the next example.

5. Take $\mathscr{C}$ to be any category with pullbacks, $\mathscr{E}$ the isomorphisms and $\mathscr{M}$ all morphisms.

6. If $(\mathscr{E}, \mathscr{M})$ is a suitable factorization system on $\mathscr{C}$ then $(\mathscr{M}^{\mathrm{op}}, \mathscr{E}^{\mathrm{op}})$ is a suitable factorization system on $\mathscr{C}^{\mathrm{op}}$.

Now we remind the reader of Lemma 2.5.9 from [16].

1.4. Lemma. *In a commutative diagram of the form*

$$\begin{tikzcd} A \arrow[r, two heads, "d"] \arrow[d, tail, "\ell"'] & B \arrow[r, tail, "i"] \arrow[d, tail, "m"] & C \arrow[d, tail, "n"] \\ X \arrow[r, two heads, "e"'] & Y \arrow[r, tail, "j"'] & Z \end{tikzcd} , \tag{1.2}$$

the horizontally pasted square is a pullback if and only if both the component squares are pullbacks.

Proof. "If" is true without any condition on the morphisms. For the converse, using SFS1, take the pullback of j and n to obtain another pastable pair of squares with the same left, right and bottom sides. The top composites are equal. By factorization system properties and SFS3, the new top is also a factorization of $i \circ d$ and thus isomorphic to the given factorization. So both of the old squares are also pullbacks. ∎

We might call the diagram (1.2) an $\mathscr{M}$-morphism of factorizations. The dual of the lemma concerns pushouts in $\mathscr{E}$-morphisms of factorizations; it therefore holds. We did not use SFS5 in proving the Lemma. However condition SFS5 does tell us that the left square of (1.2) is also a pushout when the pasted diagram is a pullback.

2. The bicategory of EM-spans

Some terminology used here, for bicategories, spans and discrete fibrations, is explained in [28]; also see the beginning of Section 3 of [26].

Let $(\mathscr{E}, \mathscr{M})$ be a suitable factorization system on the category $\mathscr{C}$.

We define a bicategory $\mathrm{Spn}(\mathscr{E}, \mathscr{M})$ with the same objects as $\mathscr{C}$. The morphisms $(d, R, m) : U \to W$ are spans $U \twoheadleftarrow R \rightarrowtail W$ in $\mathscr{C}$. The 2-cells are the usual morphisms of spans in a bicategory. Composition is the usual composition of spans; this uses conditions SFS1, SFS3 and closure of $\mathscr{E}$ under composition.

Proposition 1.2 tells us that the bicategory $\mathrm{Spn}(\mathscr{E}, \mathscr{M})$ is locally preordered.

Each $(X \xrightarrow{m} Y) \in \mathscr{M}$ gives a morphism $m_* : X \xrightarrow{(1_X, X, m)} Y$ in $\mathrm{Spn}(\mathscr{E}, \mathscr{M})$ and each $(X \xrightarrow{e} Y) \in \mathscr{E}$ gives a morphism $e^* : Y \xrightarrow{(e, X, 1_X)} X$ in $\mathrm{Spn}(\mathscr{E}, \mathscr{M})$. Write $\mathscr{M}_*$ for the class of all morphisms isomorphic to m_* for some $m \in \mathscr{M}$ and write $\mathscr{E}^*$ for the class of all morphisms isomorphic to e^* with $e \in \mathscr{E}$.

2.1. Proposition. *If* $m \in \mathscr{M}$ *then* m_* *is a discrete fibration in* $\mathrm{Spn}(\mathscr{E}, \mathscr{M})$*; that is, each functor* $\mathrm{Spn}(\mathscr{E}, \mathscr{M})(K, m_*)$ *is a discrete fibration.*

Proof. We can choose the pullback so that the functor $\mathrm{Spn}(\mathscr{E}, \mathscr{M})(K, m_*)$, up to isomorphism, takes $K \xrightarrow{(e,R,n)} X$ to $K \xrightarrow{(e,R,mn)} Y$. Given $f : (d, S, \ell) \Rightarrow (e, R, mn)$ we see that $mnf = \ell$, so $f \in \mathscr{M}$. Then $f : (d, S, nf) \Rightarrow (e, R, n)$ is the only 2-cell lifting this f. ∎

2.2. Proposition. *A morphism of* $\mathrm{Spn}(\mathscr{E},\mathscr{M})$ *has an adjoint if and only if it is an equivalence if and only if both legs of the span are invertible in* $\mathscr{C}$.

Proof. As in Proposition 2 of [7], we see that a span $X \xrightarrow{(e,R,m)} Y$ can only have a right adjoint if e is invertible and, in this case, the right adjoint must, up to isomorphism, be $Y \xrightarrow{(m,R,e)} X$. For this, we require $m \in \mathscr{E}$. So m is also invertible. Then it is clear that $X \xrightarrow{(e,R,m)} Y$ is an equivalence. ∎

Notice that 2-cells between members of $\mathscr{M}_*$, 2-cells between members of $\mathscr{E}^*$, and 2-cells from a member of $\mathscr{E}^*$ to a member of $\mathscr{M}_*$, are all invertible. In the last case, the existence of such a 2-cell implies both its domain and codomain are equivalences. A 2-cell from an $\mathscr{M}_*$ to an $\mathscr{E}_*$ requires a morphism in $\mathscr{M}$ with a left inverse in $\mathscr{E}$. The following property makes one think of a distributive law for monads and a bicategorical comma construction.

2.3. Proposition. *Given* $X \xrightarrow{m_*} Y \in \mathscr{M}_*$ *and* $Z \xrightarrow{e^*} Y \in \mathscr{E}^*$, *there exists a diagram of the form*

$$\begin{tikzcd} J \arrow[r, "\bar{e}^*"] \arrow[d, "\bar{m}_*"'] \arrow[dr, phantom, "\Leftarrow"] & X \arrow[d, "m_*"] \\ Z \arrow[r, "e^*"'] & Y \end{tikzcd} \tag{2.3}$$

in $\mathrm{Spn}(\mathscr{E},\mathscr{M})$, *with* $\bar{e}^* \in \mathscr{E}^*$ *and* $\bar{m}_* \in \mathscr{M}_*$, *which is unique up to equivalence.*

Proof. Interpreting $m_* \circ \bar{e}^* \leqslant e^* \circ \bar{m}_*$, we see that $\bar{m} \circ \bar{e}$ is forced to be an $(\mathscr{E},\mathscr{M})$ factorization of $e \circ m$. For clarification: the uniqueness clause means that, given any span $X \xleftarrow{(d,R,n)} K \xrightarrow{(c,S,\ell)} Z$ with n and c invertible, there exists an equivalence $K \xrightarrow{(s,U,t)} J$ such that $\bar{e}^* \circ (s,U,t) \cong (d,R,n)$ and $\bar{m}_* \circ (s,U,t) \cong (c,S,\ell)$, and this equivalence is unique up to compatible isomorphism. ∎

2.4. Proposition. $(\mathscr{E}^*,\mathscr{M}_*)$ *is a factorization system on the bicategory* $\mathrm{Spn}(\mathscr{E},\mathscr{M})$.

Proof. Both $\mathscr{E}^*$ and $\mathscr{M}_*$ are closed under composition and contain the equivalences. For FS2, every morphism $U \xrightarrow{(d,R,m)} W$ decomposes as $U \xrightarrow{d^*} R \xrightarrow{m_*} W$; this decomposition $\mathscr{M}_*\mathscr{E}^*$ is unique up to equivalence. For FS1, the bipullback form of (1.1) can be routinely checked. ∎

2.5. Proposition. *Pullbacks in* $\mathscr{C}$ *whose morphisms are all in* $\mathscr{M}$ *are taken by* $(-)_*$ *to bipullbacks in* $\mathrm{Spn}(\mathscr{E},\mathscr{M})$. *Also, pushouts in* $\mathscr{C}$ *whose morphisms are all in* $\mathscr{E}$ *are taken by* $(-)^*$ *to bipullbacks in* $\mathrm{Spn}(\mathscr{E},\mathscr{M})$.

3. Relations as spans of spans

By regular categories we mean those in the sense of Barr [1] which admit all finite limits. One characterization of the bicategory of relations in a regular category was given in [7]. A relation from X to Z in a regular category is a jointly monomorphic span from X to Z; these are composed using span composition followed by factorization. Equivalently, a relation from X to Z is a subobject of $X \times Z$.

The category Grp of groups is regular. So relations are subgroups of products $X \times Z$. The Goursat Lemma [14] is a bijection between subgroups $S \leqslant X \times Z$ of a cartesian product of groups X and Z and end-fixed isomorphism classes of diagrams

$$X \xleftarrow{m} U \xrightarrow{d} Y \xleftarrow{e} V \xrightarrow{n} Z \; . \tag{3.4}$$

To obtain S from (3.4), take the pullback $U \xleftarrow{\bar{e}} P \xrightarrow{\bar{d}} V$ of $U \xrightarrow{d} Y \xleftarrow{e} V$ then S is the image of $P \xrightarrow{(m\bar{e},e\bar{d})} X \times Z$. To obtain the zig-zag (3.4) from $S \hookrightarrow X \times Z$, factorize the two restricted projections to obtain

$$X \xleftarrow{m} U \xleftarrow{e'} S \xrightarrow{d'} V \xrightarrow{n} Z \; ,$$

then pushout e' and d' to obtain d and e. (Lambek [21] proved a similar result for some categories of algebraic structures other than groups; also see [8, 15] for Goursat regular categories.)

This motivates the definition of *relation* from X to Z in a category $\mathscr{C}$ equipped with a suitable factorization system $(\mathscr{E}, \mathscr{M})$ as an isomorphism class of diagrams of the form (3.4). A good reference is Grandis [16] for the case where $\mathscr{C}$ is Puppe-exact.

The starting point for the present paper was the simple observation that a relation diagram (3.4) is a span in $\mathrm{Spn}(\mathscr{E}, \mathscr{M})$:

$$(d, U, m, Y, e, V, n) \;:\; X \xleftarrow{(d,U,m)} Y \xrightarrow{(e,V,n)} Z \; . \tag{3.5}$$

We would like to define the category $\mathrm{Rel}(\mathscr{E}, \mathscr{M})$ to have equivalence classes of spans in $\mathrm{Spn}(\mathscr{E}, \mathscr{M})$ as morphisms. This is satisfactory as a definition of the underlying graph but for the composition we need a well-defined way to compose spans in $\mathrm{Spn}(\mathscr{E}, \mathscr{M})$.

4. A fake pullback construction

Let $(\mathscr{E}, \mathscr{M})$ be a suitable factorization system on a category $\mathscr{C}$. Although $\mathrm{Spn}(\mathscr{E}, \mathscr{M})$ may not have all bipullbacks, we will now show that it does allow some kind of span composition and this gives a composition of relations. The construction and proof of associativity restructures that of Calenko [6]. We will see in Section 5 that the properties of $\mathrm{Spn}(\mathscr{E}, \mathscr{M})$ established in Section 2 allow an abstract proof of associativity of composition of relations.

Take any cospan $U \xrightarrow{(d,R,m)} W \xleftarrow{(e,S,n)} V$ in $\mathrm{Spn}(\mathscr{E},\mathscr{M})$. Construct the diagram

$$\begin{tikzcd}
Q & Y \arrow[l, two heads, "s"'] \arrow[r, tail, "j"] & V \\
X \arrow[u, two heads, "r"] \arrow[d, tail, "i"'] & Z \arrow[l, two heads, "\bar{d}"] \arrow[r, tail, "\bar{m}"'] \arrow[u, two heads, "\bar{e}"] \arrow[d, tail, "\bar{n}"] & S \arrow[u, two heads, "e"'] \arrow[d, tail, "n"] \\
U & R \arrow[l, two heads, "d"] \arrow[r, tail, "m"'] & W
\end{tikzcd} \tag{4.6}$$

in which the bottom right square is a pullback of $R \rightarrowtail W \twoheadleftarrow S$, the bottom left square is an $(\mathscr{E},\mathscr{M})$-factorization of the composite $Z \rightarrowtail R \twoheadrightarrow U$, the top right square is an $(\mathscr{E},\mathscr{M})$-factorization of the composite $Z \rightarrowtail S \twoheadrightarrow V$, and the top left square is a pushout of the span $X \twoheadleftarrow Z \twoheadrightarrow Y$.

We call the span $U \xleftarrow{(r,X,i)} Q \xrightarrow{(s,Y,j)} V$ the *fake pullback* of the given cospan $U \xrightarrow{(d,R,m)} W \xleftarrow{(e,S,n)} V$. We obtain the diagram (4.7) in $\mathrm{Spn}(\mathscr{E},\mathscr{M})$. The top left square comes from a pushout in $\mathscr{C}$, the bottom right square from a pullback in $\mathscr{C}$, while the 2-cells come from factorizing an $\mathscr{M}$ followed by an $\mathscr{E}$ as an $\mathscr{E}$ followed by an $\mathscr{M}$.

$$\begin{tikzcd}
Q \arrow[r, "s^*"] \arrow[d, "r^*"'] \arrow[dr, phantom, "\lrcorner"] & Y \arrow[r, "j_*"] \arrow[d, "\bar{e}^*"] \arrow[dr, phantom, "\Rightarrow"] & V \arrow[d, "e^*"] \\
X \arrow[r, "\bar{d}^*"] \arrow[d, "i_*"'] \arrow[dr, phantom, "\Leftarrow"] & Z \arrow[r, "\bar{m}_*"] \arrow[d, "\bar{n}_*"] \arrow[dr, phantom, "\lrcorner"] & S \arrow[d, "n_*"] \\
U \arrow[r, "d^*"'] & R \arrow[r, "m_*"'] & W
\end{tikzcd} \tag{4.7}$$

4.1. Remark.

a. If d is invertible, so is s. If m is invertible, so is j.

b. If $(\mathscr{E},\mathscr{M})$ is proper (that is, every $\mathscr{E}$ is an epimorphism and every $\mathscr{M}$ is a monomorphism) then every morphism $\mathbf{r} : X \to Y$ of $\mathrm{Spn}(\mathscr{E},\mathscr{M})$ is a "fake monomorphism" in the sense that the fake pullback of $X \xrightarrow{\mathbf{r}} Y \xleftarrow{\mathbf{r}} X$ is the identity span $X \xleftarrow{1_X} X \xrightarrow{1_X} X$.

5. An abstraction

A bicategory $\mathscr{S}$ is defined to be *fake pullback ready* when it is locally preordered and is equipped with a factorization system $(\mathscr{U},\mathscr{L})$ satisfying the following conditions:

V1. bipullbacks of $\mathscr{U}$s along $\mathscr{U}$s exist and are in $\mathscr{U}$, and bipullbacks of $\mathscr{L}$s along $\mathscr{L}$s exist and are in $\mathscr{L}$;

V2. given $X \xrightarrow{a} Z \xleftarrow{x} Y$ with $a \in \mathscr{U}$ and $x \in \mathscr{L}$, there exists a square

$$\begin{tikzcd} U \arrow[r, "b"] \arrow[d, "y"'] \arrow[dr, phantom, "\Leftarrow"] & Y \arrow[d, "x"] \\ X \arrow[r, "a"'] & Z \end{tikzcd} , \tag{5.8}$$

with $b \in \mathscr{U}$ and $y \in \mathscr{L}$, which is unique up to equivalence;

V3. given a diagram

$$\begin{tikzcd} X \arrow[r, "x"] \arrow[d, "r"'] \arrow[dr, phantom, "\lrcorner"] & Y \arrow[r, "a"] \arrow[d, "s"'] \arrow[dr, phantom, "\Leftarrow"] & Z \arrow[d, "t"] \\ A \arrow[r, "y"'] & B \arrow[r, "b"'] & C \end{tikzcd} , \tag{5.9}$$

with the left square a bipullback, $r, s, t, x, y \in \mathscr{L}$ and $a, b \in \mathscr{U}$, and factorizations $a \circ x \cong v \circ c$ and $b \circ y \cong w \circ d$ with $v, w \in \mathscr{L}$ and $c, d \in \mathscr{U}$, there exists a diagram

$$\begin{tikzcd} X \arrow[r, "c"] \arrow[d, "r"'] \arrow[dr, phantom, "\Leftarrow"] & I \arrow[r, "v"] \arrow[d, "q"] \arrow[dr, phantom, "\lrcorner"] & Z \arrow[d, "t"] \\ A \arrow[r, "d"'] & J \arrow[r, "w"'] & C \end{tikzcd} \tag{5.10}$$

with the right square a bipullback and $q \in \mathscr{L}$;

V4. given a diagram

$$\begin{tikzcd} D \arrow[r, "u"] \arrow[d, "e"'] \arrow[dr, phantom, "\Rightarrow"] & E \arrow[r, "h"] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner"] & F \arrow[d, "g"] \\ X \arrow[r, "x"'] & Y \arrow[r, "a"'] & Z \end{tikzcd} \tag{5.11}$$

with the right square a bipullback, $x, u \in \mathscr{L}$ and $h, a, e, f, g \in \mathscr{U}$, and factorizations $h \circ u \cong p \circ k$ and $a \circ x \cong v \circ c$ with $v, p \in \mathscr{L}$ and $c, k \in \mathscr{U}$, there exists a diagram

$$\begin{tikzcd} D \arrow[r, "k"] \arrow[d, "e"'] \arrow[dr, phantom, "\lrcorner"] & K \arrow[r, "p"] \arrow[d, "j"] \arrow[dr, phantom, "\Rightarrow"] & F \arrow[d, "g"] \\ X \arrow[r, "c"'] & I \arrow[r, "v"'] & Z \end{tikzcd} \tag{5.12}$$

with the left square a bipullback and $j \in \mathscr{U}$.

5.1. Proposition. *Let $(\mathscr{E}, \mathscr{M})$ be a suitable factorization system on the category $\mathscr{C}$. The locally preordered bicategory* $\mathrm{Spn}(\mathscr{E}, \mathscr{M})$ *is rendered fake pullback ready by the factorization system $(\mathscr{E}^*, \mathscr{M}_*)$ of Proposition 2.4.*

PROOF. Condition V1 is provided by Proposition 2.5. Condition V2 is provided by Proposition 2.3. For condition V3, consider diagram (5.9) with $x_*, a^*, \dots$ replacing $x, a, \dots$ since $\mathscr{L} = \mathscr{M}_*$ and $\mathscr{U} = \mathscr{E}^*$ in this case. The left square amounts to the pullback shown as the right-hand square on the left-hand side of (5.13). The right-hand square with the 2-cell amounts to the factorization $s \circ a = b \circ t$. Now form the pullback on the left of the left-hand side of (5.13) and the pullback on the right of the right-hand side of (5.13). Since $btv = sav = sxc = yrc$, there exists a unique q such that $dq = rc$ and $wq = tv$. So we have the equal pastings as shown in (5.13).

$$\begin{tikzcd} I \arrow[r, tail, "c"] \arrow[d, tail, "v"'] \arrow[dr, phantom, "\lrcorner"] & X \arrow[r, two heads, "r"] \arrow[d, tail, "x"'] \arrow[dr, phantom, "\lrcorner"] & A \arrow[d, tail, "y"] \\ Z \arrow[r, tail, "a"'] & Y \arrow[r, two heads, "s"'] & B \end{tikzcd} \quad = \quad \begin{tikzcd} I \arrow[r, two heads, "q"] \arrow[d, tail, "v"'] & J \arrow[r, tail, "d"] \arrow[d, tail, "w"'] \arrow[dr, phantom, "\lrcorner"] & A \arrow[d, tail, "y"] \\ Z \arrow[r, two heads, "t"'] & C \arrow[r, tail, "b"'] & B \end{tikzcd} \tag{5.13}$$

It follows that the left diagram on the right-hand side of (5.13) is a pullback and, by SFS3, that $q \in \mathscr{E}$. Diagram (5.10) results.

It is V4 which requires suitable factorization condition SFS5. Consider diagram (5.11). We have the pushout on the right of the left-hand side of (5.14) and the factorization $fx = ue$. Form the pullback of a and x and note, using one direction of SFS5, that it gives the pushout on the left of the left-hand side of (5.14). Next, factorize $gv = pj$ through K with $p \in \mathscr{M}$ and $j \in \mathscr{E}$. Using functoriality of factorization FS1, we obtain a unique $k : K \to D$ with $kj = ec$ and $uk = hp$.

$$\begin{tikzcd} I \arrow[r, tail, "v"] \arrow[d, two heads, "c"'] \arrow[dr, phantom, "\ulcorner"] & Z \arrow[r, two heads, "g"] \arrow[d, two heads, "a"'] \arrow[dr, phantom, "\ulcorner"] & F \arrow[d, two heads, "h"] \\ X \arrow[r, tail, "x"'] & Y \arrow[r, two heads, "f"'] & E \end{tikzcd} \quad = \quad \begin{tikzcd} I \arrow[r, two heads, "j"] \arrow[d, two heads, "c"'] & K \arrow[r, tail, "p"] \arrow[d, two heads, "k"] & F \arrow[d, two heads, "h"] \\ X \arrow[r, two heads, "e"'] & D \arrow[r, tail, "u"'] & E \end{tikzcd} \tag{5.14}$$

By the dual of Lemma 1.4, it follows that both squares on the right-hand side of (5.14) are pushouts. Diagram (5.12) results using the other direction of SFS5 to see that the right square on the right-hand side of (5.14) is a pullback and hence $p_* k^* = h^* u_*$. ∎

Let $\mathscr{S}$ be fake pullback ready. The *fake pullback* of a cospan $U \xrightarrow{r} W \xleftarrow{s} V$ in $\mathscr{S}$ is constructed as follows. Factorize $r \cong x \circ a$ and $s \cong y \circ b$ with $a, b \in \mathscr{U}$ and $x, y \in \mathscr{L}$. Using half of V1, take the bipullback of x and y as shown in the bottom right square of (5.15). Now construct the bottom left and top right squares of (5.15) using V2. Using the other half of V1, we obtain the top left bipullback.

$$\begin{tikzcd} Q \arrow[r, "a'"] \arrow[d, "b'"'] \arrow[dr, phantom, "\lrcorner"] & Y \arrow[r, "\bar{x}"] \arrow[d, "\bar{b}"] \arrow[dr, phantom, "\Rightarrow"] & V \arrow[d, "b"] \\ X \arrow[r, "\bar{a}"] \arrow[d, "\bar{y}"'] \arrow[dr, phantom, "\Leftarrow"] & Z \arrow[r, "x'"] \arrow[d, "y'"] \arrow[dr, phantom, "\lrcorner"] & S \arrow[d, "y"] \\ U \arrow[r, "a"'] & R \arrow[r, "x"'] & W \end{tikzcd} \tag{5.15}$$

The span $U \xleftarrow{\bar{y}b'} Q \xrightarrow{\bar{x}a'} V$ is our fake pullback of $U \xrightarrow{r} W \xleftarrow{s} V$.

5.2. Proposition. *Fake pullbacks are symmetric. That is, if $U \xleftarrow{\bar{s}} Q \xrightarrow{\bar{r}} V$ is a fake pullback of $U \xrightarrow{r} W \xleftarrow{s} V$ then $V \xleftarrow{\bar{r}} Q \xrightarrow{\bar{s}} U$ is a fake pullback of $V \xrightarrow{s} W \xleftarrow{r} U$.*

Proof. In (5.15), the bipullbacks are symmetric and both 2-cells point to the boundary of the diagram. So the diagram is symmetric about its main diagonal. ∎

Note that, should a bipullback

$$\begin{tikzcd}
U \arrow[r, "b"] \arrow[d, "y"'] \arrow[dr, phantom, "\overset{\Leftarrow}{\cong}"] & Y \arrow[d, "x"] \\
X \arrow[r, "a"'] & Z
\end{tikzcd},$$

of $X \xrightarrow{a} Z \xleftarrow{x} Y$ exist with $y \in \mathscr{L}$ and $b \in \mathscr{U}$, it would provide the square for V2. This happens for example when a is an identity, b is an identity, and $y = x$. Consequently:

5.3. Proposition. *An identity morphism provides a fake pullback of an identity morphism along any morphism.*

$$\begin{tikzcd}
P \arrow[r, "\bar{t}"] \arrow[d, "\bar{\bar{s}}"'] \arrow[dr, phantom, "\text{fakepb}"] & Q \arrow[r, "\bar{r}"] \arrow[d, "\bar{s}"] \arrow[dr, phantom, "\text{fakepb}"] & V \arrow[d, "s"] \\
X \arrow[r, "t"'] & U \arrow[r, "r"'] & W
\end{tikzcd}
\qquad
\begin{tikzcd}
P \arrow[r, "\bar{r}\bar{t}"] \arrow[d, "\bar{\bar{s}}"'] \arrow[dr, phantom, "\text{fakepb}"] & V \arrow[d, "s"] \\
X \arrow[r, "rt"'] & W
\end{tikzcd}
\tag{5.16}$$

5.4. Proposition. *Fake pullbacks stack. That is, if the two squares on the left of (5.16) are fake pullbacks then so is the pasted square on the right of (5.16).*

Proof. Faced with a diagram like

$$\begin{tikzcd}
Q \arrow[r, "u"] \arrow[d, "u"] \arrow[dr, phantom, "\lrcorner"] & Y \arrow[r, "\ell"] \arrow[d, "u"] \arrow[dr, phantom, "\Rightarrow"] & V \arrow[r, "u"] \arrow[d, "u"] \arrow[dr, phantom, "\lrcorner"] & A \arrow[r, "\ell"] \arrow[d, "u"] \arrow[dr, phantom, "\Rightarrow"] & D \arrow[d, "u"] \\
X \arrow[r, "u"] \arrow[d, "\ell"] \arrow[dr, phantom, "\Leftarrow"] & Z \arrow[r, "\ell"] \arrow[d, "\ell"] \arrow[dr, phantom, "\lrcorner"] & S \arrow[r, "u"] \arrow[d, "\ell"] \arrow[dr, phantom, "\Leftarrow"] & B \arrow[r, "\ell"] \arrow[d, "\ell"] \arrow[dr, phantom, "\lrcorner"] & E \arrow[d, "\ell"] \\
U \arrow[r, "u"] & R \arrow[r, "\ell"] & W \arrow[r, "u"] & C \arrow[r, "\ell"] & F
\end{tikzcd}$$

in which the arrows marked u are in $\mathscr{U}$ and those marked ℓ are in $\mathscr{L}$, we apply condition V3 to the middle bottom two squares and condition V4 to the middle top two squares to obtain

$$\begin{tikzcd}
Q \arrow[r, "u"] \arrow[d, "u"] \arrow[dr, phantom, "\lrcorner"] & Y \arrow[r, "u"] \arrow[d, "u"] \arrow[dr, phantom, "\lrcorner"] & I \arrow[r, "\ell"] \arrow[d, "u"] \arrow[dr, phantom, "\Rightarrow"] & A \arrow[r, "\ell"] \arrow[d, "u"] \arrow[dr, phantom, "\Rightarrow"] & D \arrow[d, "u"] \\
X \arrow[r, "u"] \arrow[d, "\ell"] \arrow[dr, phantom, "\Leftarrow"] & Z \arrow[r, "u"] \arrow[d, "\ell"] \arrow[dr, phantom, "\Leftarrow"] & P \arrow[r, "\ell"] \arrow[d, "\ell"] \arrow[dr, phantom, "\lrcorner"] & B \arrow[r, "\ell"] \arrow[d, "\ell"] \arrow[dr, phantom, "\lrcorner"] & E \arrow[d, "\ell"] \\
U \arrow[r, "u"] & R \arrow[r, "u"] & P' \arrow[r, "\ell"] & C \arrow[r, "\ell"] & F
\end{tikzcd}$$

which is again a fake pullback. ■

As a corollary of all this we have:

5.5. Theorem. *Let $\mathscr{S}$ be a fake pullback ready bicategory. There is a category* $\mathrm{Spn}[\mathscr{S}]$ *whose objects are those of $\mathscr{S}$, whose morphisms are equivalence classes of spans in $\mathscr{S}$, and whose composition is defined by fake pullback.*

Applying this theorem to the bicategory $\mathscr{S} = \mathrm{Spn}(\mathscr{E}, \mathscr{M})$ of Proposition 5.1, we obtain our category $\mathrm{Spn}[\mathscr{S}] = \mathrm{Rel}(\mathscr{E}, \mathscr{M})$ of relations in $\mathscr{C}$ for the factorization system $(\mathscr{E}, \mathscr{M})$.

5.6. Remark. Given Remark 4.1, we might call $\mathscr{S}$ *proper* when the identity span provides a fake pullback of each morphism with itself. In this case, each morphism $X \xrightarrow{\mathbf{r}} Y$ in $\mathrm{Spn}[\mathscr{S}]$ satisfies $\mathbf{r}\mathbf{r}^\circ\mathbf{r} = \mathbf{r}$ where $\mathbf{r}^\circ : Y \to X$ is the reverse span of $\mathbf{r}$.

6. A distributive law formalism and the PROP example

The referee of this paper suggested explaining more about how the PROP for monoids fits into the fake pullback format and mentioned the Rosebrugh-Wood paper [25]. That paper showed the connection between factorizations and distributive laws (in the sense of Beck [2]), and showed that pullbacks are generalized distributive laws. The referee thought it likely that fake pullbacks would provide distributive laws too. This section goes some way towards addressing these points.

We work in any monoidal category $\mathscr{V}$. For any monoid M we will denote the multiplication by $\mu : M \otimes M \to M$ and the unit by $\eta : I \to M$. Let $\mathscr{V}[M]$ denote the monoidal category with objects (X, ϕ) where X and $\phi : M \otimes X \to X \otimes M$ are in $\mathscr{V}$, where a morphism $f : (X, \phi) \to (Y, \psi)$ is a morphism $f : X \to Y$ in $\mathscr{V}$ such that $(f \otimes 1_M)\phi = \psi(1_M \otimes f)$, and tensor product is defined by $(X, \phi) \otimes (X', \phi') = (X \otimes X', (1_X \otimes \psi)(\phi \otimes 1_{X'})$. A *distributive law* in the monoidal category $\mathscr{V}$ of a monoid M around a monoid E is a morphism $\lambda : E \otimes M \to M \otimes E$ satisfying the four conditions as in (6.17) and (6.18). This amounts to saying that (E, λ) is a monoid in $\mathscr{V}[M]$.

$$\begin{tikzcd} E\otimes E\otimes M \arrow[r, "1\otimes\lambda"] \arrow[d, "\mu\otimes 1"'] & E\otimes M\otimes E \arrow[r, "\lambda\otimes 1"] & M\otimes E\otimes E \arrow[d, "1\otimes\mu"] \\ E\otimes M \arrow[rr, "\lambda"'] & & M\otimes E \end{tikzcd} \qquad \begin{tikzcd} M \arrow[d, "\eta 1"'] \arrow[dr, "1\eta"] & \\ E\otimes M \arrow[r, "\lambda"'] & M\otimes E \end{tikzcd} \tag{6.17}$$

$$\begin{tikzcd} E\otimes M\otimes M \arrow[r, "\lambda\otimes 1"] \arrow[d, "1\otimes\mu"'] & M\otimes E\otimes M \arrow[r, "1\otimes\lambda"] & M\otimes M\otimes E \arrow[d, "\mu\otimes 1"] \\ E\otimes M \arrow[rr, "\lambda"'] & & M\otimes E \end{tikzcd} \qquad \begin{tikzcd} E \arrow[d, "1\eta"'] \arrow[dr, "\eta 1"] & \\ E\otimes M \arrow[r, "\lambda"'] & M\otimes E \end{tikzcd} \tag{6.18}$$

Then we obtain a monoid $S = M \otimes E$, with multiplication

$$M \otimes E \otimes M \otimes E \xrightarrow{1\otimes\lambda\otimes 1} M \otimes M \otimes E \otimes E \xrightarrow{\mu\otimes\mu} M \otimes E$$

and unit $I \xrightarrow{\eta\otimes\eta} M \otimes E$, called the *wreath product* of M and E (after [20]).

In Proposition 6.1 we describe distributive laws in the monoidal category $\mathscr{V}[M][(E,\lambda)]$. Further iteration of this process is also of interest as studied by Cheng [9].

6.1. Proposition. *Given four monoids E, M, E' and M', and six distributive laws*

$$E \otimes M \xrightarrow{\lambda} M \otimes E \qquad M' \otimes E' \xrightarrow{\lambda'} E' \otimes M' \qquad E' \otimes E \xrightarrow{\pi} E \otimes E'$$
$$M' \otimes M \xrightarrow{\kappa} M \otimes M' \qquad M' \otimes E \xrightarrow{\nu} E \otimes M' \qquad E' \otimes M \xrightarrow{\nu'} M \otimes E' \tag{6.19}$$

satisfying the four hexagonal conditions (6.20), (6.21), (6.22), (6.23) *(in which the symbol $\otimes$ is omitted to save space), let $S = M \otimes E$ and $S' = E' \otimes M'$ be the wreath product monoids for λ and λ'. Then the composite*

$$\sigma = (E'M'ME \xrightarrow{1\otimes\kappa\otimes 1} E'MM'E \xrightarrow{\nu'\otimes\nu} ME'EM' \xrightarrow{1\otimes\pi\otimes 1} MEE'M')$$

is a distributive law of S around S'.

$$\begin{tikzcd} M'EM \arrow[r, "\nu 1"] \arrow[d, "1\lambda"'] & EM'M \arrow[r, "1\kappa"] & EMM' \arrow[d, "\lambda 1"] \\ M'ME \arrow[r, "\kappa 1"'] & MM'E \arrow[r, "1\nu"'] & MEM' \end{tikzcd} \tag{6.20}$$

$$\begin{tikzcd} E'EM \arrow[r, "1\lambda"] \arrow[d, "\pi 1"'] & E'ME \arrow[r, "\nu' 1"] & ME'E \arrow[d, "1\pi"] \\ EE'M \arrow[r, "1\nu'"'] & EME' \arrow[r, "\lambda 1"'] & MEE' \end{tikzcd} \tag{6.21}$$

$$\begin{tikzcd} M'E'E \arrow[r, "\lambda' 1"] \arrow[d, "1\pi"'] & E'M'E \arrow[r, "1\nu"] & E'EM' \arrow[d, "\pi 1"] \\ M'EE' \arrow[r, "\nu 1"'] & EM'E' \arrow[r, "1\lambda'"'] & EE'M' \end{tikzcd} \tag{6.22}$$

$$\begin{tikzcd} M'E'M \arrow[r, "1\nu'"] \arrow[d, "\lambda' 1"'] & M'ME \arrow[r, "\kappa 1"] & MM'E' \arrow[d, "1\lambda'"] \\ E'M'M \arrow[r, "1\kappa"'] & E'MM' \arrow[r, "\nu' 1"'] & ME'M' \end{tikzcd} \tag{6.23}$$

PROOF. We must prove that σ satisfies conditions (6.17), (6.18) while remaining aware that the multiplications for S and S' involve λ and λ'. To my mind the best way to do this is using string diagrams for monoidal categories as per [17]. Multiplications of monoids are depicted as "Y"-shaped string diagrams while units are lollipops. The distributive laws are depicted by two strings crossing left over right. Diagrams (6.20), (6.21), (6.22), (6.23) appear as Reidemeister moves of Type III (or, equally as Yang-Baxter or braid equations). The diagram for σ is four strings with the first two crossed over the second two. With this, we leave the string calculations to the reader. ∎

The particular monoidal category $\mathscr{V}$ to which we wish to apply this proposition is the category $[\Omega \times \Omega, \mathrm{Set}]$ of families $A = (A(x,y))_{x,y\in\Omega}$ of sets doubly indexed by the set Ω, with tensor product defined by

$$(B \otimes A)(x,z) = \sum_{z\in\Omega} B(y,z) \times A(x,y) \ .$$

Monoids M in this $\mathscr{V}$ are categories with Ω as set of objects. This $\mathscr{V}$ also has a tensor-product-reversing involution $(-)' : \mathscr{V}^{\mathrm{rev}} \to \mathscr{V}$ given by the transpose $A'(x,y) = A(y,x)$. For a monoid M, the monoid M' is the opposite of M as categories.

Now, to have the data for Proposition 6.1, we only require two monoids M and E, the other two obtained by applying the involution, and four distributive laws λ, π, κ and ν, the other two obtained by applying the involution to λ and ν, subject to the conditions $\pi' = \pi$, $\kappa' = \kappa$, (6.20) and (6.21).

From [25] we know that monoids E and M with a distributive law $E \otimes M \xrightarrow{\lambda} M \otimes E$ amount to a "strict factorization system" (E, M) on the category $C = M \otimes E$ given by the wreath product. As pointed out in [25], this terminology from [18] is not perfect since strict factorization systems need not be factorization systems; neither E nor M is required to contain all the isomorphisms. Adaptations of "distributive law" required to obtain factorization systems themselves are provided in [25] and [20]; possibly [27] would also be of interest in this connection.

Moreover, Rosebrugh-Wood [25] observe that pullbacks in a category C also provide an example of their generalized distributive laws. In our present notation, for a monoid C in $\mathscr{V} = [\Omega\times\Omega, \mathrm{Set}]$, chosen pullback in the category C would be a morphism $C'\otimes C \to C\otimes C'$. Any generalized distributive law of the form $C'\otimes C \to C\otimes C'$ could be considered a formal fake pullback which turns the formal wreath product $C \otimes C'$ into a category of spans in C.

To accommodate the setting of Section 5 we need a generalized form of distributive law. Then, suffice it to say here that λ comes from the factorization system $(\mathscr{U}, \mathscr{L})$ on $\mathscr{S}$, that π and κ come from condition V1, that ν comes from condition V2, while the hexagons (6.20) and (6.21) amount to conditions V3 and V4; and of course σ is the fake pullback.

Let us now look at the PROP for monoids; for background see [22, 23, 24]. Take Ω to be the set $\mathbb{N}$ of natural numbers $n \geqslant 0$. Write $\langle n\rangle$ for the set $\{1, 2, \ldots, n\}$. Write $\mathbb{P}$ for the permutation category; that is, its set of objects is $\mathbb{N}$ while the homset $\mathbb{P}(m, n)$ is empty unless $m = n$ and $\mathbb{P}(n, n)$ is the set of bijective functions $\alpha : \langle n\rangle \to \langle n\rangle$. Write Δ for the algebraists' simplicial category; the set of objects is again $\mathbb{N}$ while the homset $\Delta(m, n)$ is the set of order-preserving functions $\xi : \langle m\rangle \to \langle n\rangle$.

A distributive law $\lambda : \mathbb{P} \otimes \Delta \to \Delta \otimes \mathbb{P}$ is described, for example, in Section 4 of [10] where the notation is:

$$\lambda(n \xrightarrow{\alpha} n, m \xrightarrow{\xi} n) = (m \xrightarrow{\xi_\alpha} n, m \xrightarrow{\alpha^\xi} m)\ .$$

Actually, the distributive law is described explicitly there, both categorically and geometrically[1], for the braid category $\mathbb{B}$ rather than $\mathbb{P}$. The wreath product $\mathbb{M} = \Delta \otimes \mathbb{P}$ is the category underlying the PROP for monoids. Here we are not discussing the symmetric monoidal structure on $\mathbb{M}$; however, see Section 5 of Lack [19].

A morphism of $\mathbb{M}$ is a pair $(\xi, \alpha) : m \to n$. However, we can identify such morphisms with functions $f : \langle m\rangle \to \langle n\rangle$ equipped with a linear order on each fibre $f^{-1}(j)$. For, the composite function $f = \xi\alpha$ has a linear order on each fibre $f^{-1}(j)$ defined by $i \leqslant_j i'$ if and only if $\alpha(i) \leqslant \alpha(i')$ in $\langle m\rangle$. Conversely, each function $f : \langle m\rangle \to \langle n\rangle$ equipped with a linear order $\leqslant_j$ on each fibre arises in that way from a unique pair (ξ, α) with $\xi : m \to n$ in Δ and $\alpha : m \to m$ in $\mathbb{P}$. The composition of $\mathbb{M}$ using the distributive law λ above reinterprets as composition of functions $\langle m\rangle \xrightarrow{f} \langle n\rangle \xrightarrow{g} \langle p\rangle$ with linear order on each fibre

$$(gf)^{-1}(k) = \sum_{j \in g^{-1}(k)} f^{-1}(j)$$

defined by the usual ordered sum of ordered sets.

While $\mathbb{M}$ does not have pullbacks, we do have a distributive law $\sigma : \mathbb{M}' \otimes \mathbb{M} \to \mathbb{M} \otimes \mathbb{M}'$. It does not come from component distributive laws as in Proposition 6.1. To define this σ, take the pullback span $\langle m\rangle \xleftarrow{\bar{g}} \langle q\rangle \xrightarrow{\bar{f}} \langle p\rangle$ of the cospan $\langle m\rangle \xrightarrow{f} \langle n\rangle \xleftarrow{g} \langle p\rangle$ in the skeletal category of finite sets. We use the canonical isomorphisms

$$\bar{f}^{-1}(k) \cong f^{-1}(g(k))\ , \qquad \bar{g}^{-1}(i) \cong g^{-1}(f(i))$$

to transport linear orders of the fibres of f and g to those of $\bar{f}$ and $\bar{g}$. In this way, we have $\sigma(n \xleftarrow{(\zeta,\beta)} p, m \xrightarrow{(\xi,\alpha)} n) = (q \xrightarrow{(\bar{\xi},\bar{\alpha})} p, m \xleftarrow{(\bar{\zeta},\bar{\beta})} q)$ such that the square

$$\begin{tikzcd}
q \arrow[r, "(\bar{\xi},\bar{\alpha})"] \arrow[d, "(\bar{\zeta},\bar{\beta})"'] & p \arrow[d, "(\zeta,\beta)"] \\
m \arrow[r, "(\xi,\alpha)"'] & n
\end{tikzcd}$$

commutes in $\mathbb{M}$. The wreath product $\mathbb{M} \otimes \mathbb{M}'$ for σ is the category underlying the PROP for bimonoids (also called "bialgebras"); see Pirashvili [24].

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[1] In the diagram on top of page 59 of [10] there is an isolated dot missing on the left of the codomain on the right-hand side.

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