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A REMARKABLE ASPECT OF INTERNAL GROUPOIDS IN REGULAR MAL'TSEV CATEGORIES

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Abstract. In any finitely cocomplete regular Mal'tsev category $\mathbb{E}$, we characterize by a very simple property in $\mathbb{E}$ the cocartesian functors in the category $Grd\mathbb{E}$ of internal groupoids in $\mathbb{E}$, provided that they are defined above regular epimorphisms.

Introduction

This note is devoted to showing that in a regular Mal'tsev category $\mathbb{E}$ with pushouts of split monomorphisms, given any internal groupoid $X_\bullet$, the pushout of the split monomorphism $s_0^{X_\bullet}$ along f:

$$\begin{tikzcd} X_1 \arrow[r, two heads, "f_1"] \arrow[d, shift right=4, "d_0^{X_\bullet}"'] \arrow[d, shift left=4, "d_1^{X_\bullet}"] & Y_1 \arrow[d, dotted, shift right=4, "d_0"'] \arrow[d, dotted, shift left=4, "d_1"] \\ X_0 \arrow[r, two heads, "f"'] \arrow[u, "s_0^{X_\bullet}"] & Y \arrow[u, "s_0"] \end{tikzcd}$$

produces on the right hand side a reflexive graph which is underlying a groupoid structure as soon as the map f is a regular epimorphism. It is a consequence of the fact that, given any regular Mal'tsev category $\mathbb{E}$, the category $Grd\mathbb{E}$ of internal groupoids in $\mathbb{E}$ is a fully faithful Birkhoff subcategory of the category $RGh\mathbb{E}$ of the internal reflexive graphs in $\mathbb{E}$. It is then clear that the previous diagram provides, in a very simple way, the cocartesian internal functor above the regular epimorphism f with respect to the forgetful functor $(\ )_0 : Grd\mathbb{E} \to \mathbb{E}$.

1. Fibration of points and internal groupoids

1.1. The fibration of points. In this article, any category $\mathbb{E}$ will be supposed finitely complete. Given any map $f : X \to Y$, we use the following simplicial notations for its

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kernel equivalence relation $R[f]$:

$$\begin{tikzcd} R_2[f] \arrow[r, shift left=2, "d_2^f"] \arrow[r, "d_1^f" description] \arrow[r, shift right=2, "d_0^f"'] & R[f] \arrow[r, shift left=2, "d_1^f"] \arrow[r, shift right=2, "d_0^f"'] & X \arrow[l, "s_0^f" description] \arrow[r, "f"'] & Y \end{tikzcd}$$

and more generally for any internal category as well:

$$X_\bullet : \quad \begin{tikzcd} X_1 \times_0 X_1 \arrow[r, shift left=2, "d_2^{X_\bullet}"] \arrow[r, "d_1^{X_\bullet}" description] \arrow[r, shift right=2, "d_0^{X_\bullet}"'] & X_1 \arrow[r, shift left=2, "d_1^{X_\bullet}"] \arrow[r, shift right=2, "d_0^{X_\bullet}"'] & X_0 \arrow[l, "s_0^{X_\bullet}" description] \end{tikzcd}$$

We denote by $Pt\mathbb{E}$ the category whose objects are the split epimorphisms in $\mathbb{E}$ with a given splitting and morphisms the commutative squares between these data, and by $\P_{\mathbb{E}} : Pt\mathbb{E} \to \mathbb{E}$ the functor associating its codomain with any split epimorphism. This functor is left exact and a fibration whose cartesian maps are the pullbacks of split epimorphisms; it is called the *fibration of points* [4]. More precisely it is a *fibered reflection* [3], in the sense that it admits a fully faithful right adjoint I defined by $I(Y) = (1_Y, 1_Y)$. The fibre above an object $Y \in \mathbb{E}$ is thus pointed, and denoted by $Pt_Y\mathbb{E}$.

1.2. The groupoid monad. There is on $Pt\mathbb{E}$ a left exact monad (T, λ, μ) defined by the following diagram:

$$\begin{tikzcd} X \arrow[r, tail, "s_1"] \arrow[d, shift right, "f"'] & R[f] \arrow[d, shift right, "d_0^f"'] & R^2[f] \arrow[l, "d_2^f"'] \arrow[d, shift right, "d_0^f"'] \\ Y \arrow[r, tail, "s"'] \arrow[u, shift right, "s"'] & X \arrow[u, shift right, "s_0^f"'] & R[f] \arrow[l, "d_1^f"] \arrow[u, shift right, "s_0^f"'] \end{tikzcd}$$

$$\begin{tikzcd} (f,s) \arrow[r, tail, "\lambda_{(f,s)}"'] & T(f,s) & T^2(f,s) \arrow[l, "\mu_{(f,s)}"] \end{tikzcd}$$

1.3. Theorem. *[3] The category Alg^T is the category $Grd\mathbb{E}$ of internal groupoids in $\mathbb{E}$. The forgetful functor $(\ )_0 : Alg^T = Grd\mathbb{E} \to \mathbb{E}$ is a fibered reflection whose fully faithful right adjoint $\nabla : \mathbb{E} \to Grd\mathbb{E}$ is given by the indiscrete equivalence relations. Accordingly the* cartesian map *in $Grd\mathbb{E}$ above $f : X \to Y_0$ with codomain $Y_\bullet$ is given by the following pullback in $\mathbb{E}$:*

$$\begin{tikzcd} X_1 \arrow[r, "f_1"] \arrow[d, "(d_0^{X_\bullet}, d_1^{X_\bullet})"'] & Y_1 \arrow[d, "(d_0^{Y_\bullet}, d_1^{Y_\bullet})"] \\ X \times X \arrow[r, "f \times f"'] & Y_0 \times Y_0 \end{tikzcd}$$

So, a groupoid structure on any right hand side reflexive graph:

$$\begin{tikzcd} R_2[d_0] \arrow[r, shift left=2, "d_2^{d_0}"] \arrow[r, "d_1^{d_0}" description] \arrow[r, shift right=2, "d_0^{d_0}"'] \arrow[r, bend left=60, dotted, "R(d_2)"] & R[d_0] \arrow[r, shift left, "d_1^{d_0}" description] \arrow[r, shift right=2, "d_0^{d_0}"'] \arrow[r, bend left=60, dotted, "d_2"] & X_1 \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "d_0"'] & X_0 \arrow[l, "s_0" description] \end{tikzcd}$$

is given by a map d_2 such that the previous diagram satisfies all the simplicial identities, which, by the way, makes any commutative square in this diagram a pullback. In set-theoretical terms, the map d_2 is defined in the following way: starting with a pair (ϕ, ψ) of arrows with same domain in a given groupoid:

$$\begin{tikzcd} & \bullet \arrow[dr, dotted, "\psi.\phi^{-1}"] & \\ \bullet \arrow[ur, "\phi"] \arrow[rr, "\psi"'] & & \bullet \end{tikzcd}$$

the value of d_2 is given by the dotted arrow. Warning: in a group $(G, \circ, 1)$, the multiplication $(a, b) \mapsto a \circ b$ (written in the opposite way to the composition in a groupoid) makes this map d_2:

$$\begin{tikzcd} & \bullet \arrow[dr, dotted, "a^{-1} \circ b"] & \\ \bullet \arrow[ur, "a"] \arrow[rr, "b"'] & & \bullet \end{tikzcd}$$

be the opsubtraction ∂_G associated with the multiplication $\circ$.

2. Mal'tsev and unital categories

Mimicking what happened in Universal Algebra for the Mal'tsev varieties [16], the Mal'tsev categories were introduced in [10] and [11] as those categories in which *any reflexive relation is necessarily an equivalence relation*. One of the main properties of this kind of category is that it fits perfectly with the notion of centralization of equivalence relations as it is the case for the Mal'tsev varieties, see [19]. For that we need the notion of connector, see [7] (and also [18] and [11]):

2.1. Definition. *Let R and S be two equivalence relations on a same object X in $\mathbb{E}$, a* connector *between R and S is a morphism $p \colon R \times_X S \to X$ where $R \times_X S$ is defined by the following pullback:*

$$\begin{tikzcd} R \times_X S \arrow[r, "p_1^S"] \arrow[d, "p_0^R"'] & S \arrow[d, "d_0^S"] \\ R \arrow[r, "d_1^R"'] & X \end{tikzcd}$$

satisfying the following axioms:

1. $xSp(xRySz)$ and $p(xRySz)Rz$;

$$\begin{tikzcd}
x \arrow[rr, dash, "R"] \arrow[dr, dash, "S"'] & & y \arrow[dr, dash, "S"] & \\
& p(xRySz) \arrow[rr, dash, "R"'] & & z
\end{tikzcd}$$

2. p is a partial Mal'tsev operation, i.e. $p(xRxSy) = y$ and $p(xRySy) = x$;

3. p is left associative, i.e. $p(p(xRySz)RzSw) = p(xRySw)$ and right associative, i.e. $p(xRySp(yRzSw)) = p(xRzSw)$.

Now, another possible definition of an internal groupoid structure on a given reflexive graph is the data of a connector p between $R[d_0]$ and $R[d_1]$, see [13]. The fundamental fact for the Mal'tsev categories is the following:

2.2. Proposition. [7] *In a Mal'tsev category $\mathbb{E}$, given any pair (R, S) of equivalence relations, there is at most one connector between them, and the axiom 2 implies the two others. In this case, the equivalence relations R and S are said to* centralize each other *and this situation is denoted by $[R, S] = 0$. An equivalence relation R is said to be* abelian *when $[R, R] = 0$.*

This observation implies in particular that:
1) on any reflexive graph there is at most one groupoid structure and the forgetful functor $Grd\mathbb{E} \to Grh\mathbb{E}$ is actually fully faithful;
2) on any object X, there is at most one internal Mal'tsev operation which is then necessarily autonomous; when such an operation does exist, the object X is said to be *affine*. A collateral effect is that, in a Mal'tsev category, there are no other internal categories but the groupoids [11].

Similarly, mimicking what happened in Universal Algebra for the Jónsson-Tarski varieties [15], the unital categories where introduced in [5] as those pointed categories in which, given any pair (X, Y) of objects, the following associated pair of inclusions:

$$\begin{tikzcd}
X \arrow[r, tail, "(1_X, 0)"] & X \times Y & Y \arrow[l, tail, "(0, 1_Y)"']
\end{tikzcd}$$

is jointly strongly epic, which means that any commutative diagram of split epimorphisms

$$\begin{tikzcd}
W \arrow[r, shift left, "g"] \arrow[d, shift right, "f"'] & Y \arrow[l, shift left, "t"] \arrow[d, shift right, "\tau_Y"'] \\
X \arrow[u, shift right, "s"'] \arrow[r, shift left, "\tau_X"] & 1 \arrow[l, shift left, "0_X"] \arrow[u, shift right, "0_Y"']
\end{tikzcd}$$

determines a strongly epic factorization $W \to X \times Y$. Here, the fundamental fact is the following:

2.3. Proposition. *In a unital category $\mathbb{E}$, given any pair (f, g) of morphisms with same codomain, there is at most one map ϕ making the following triangle commute:*

$$\begin{tikzcd} X \times Y \arrow[dr, dotted, "\phi"] & Y \arrow[l, "{(0,1_Y)}"'] \arrow[d, "g"] \\ X \arrow[u, "{(1_X,0)}"] \arrow[r, "f"'] & Z \end{tikzcd}$$

This map ϕ is called the *cooperator* of the pair which is then said to commute when this ϕ does exist. An object X in $\mathbb{E}$ is said to be *commutative* when the pair $(1_X, 1_X)$ commutes.

2.4. Recalls about characterizations. In [5], the two following characterizations were asserted:

2.5. Proposition. *A category $\mathbb{E}$ is a Mal'tsev one if and only any fiber $Pt_Y\mathbb{E}$ of the fibration of points is unital.*

Then two equivalence relations (R, S) on X centralize each other if and only the two following subobjects commute in the unital fiber $Pt_X\mathbb{E}$:

$$\begin{tikzcd} R \arrow[r, tail, "{(d_0^R,d_1^R)}"] \arrow[dr, shift left, "d_0^R"'] & X \times X \arrow[d, shift right, "p_0^X"'] & S \arrow[l, tail, "{(d_0^S,d_1^S)}"'] \arrow[dl, shift right, "d_0^S"] \\ & X \arrow[ul, shift left, "s_0^R"] \arrow[u, shift right, "s_0^X"'] \arrow[ur, shift right, "s_0^S"'] & \end{tikzcd}$$

2.6. Proposition. *A category $\mathbb{E}$ is a Mal'tsev one if and only if any subreflexive graph of an internal groupoid $Y_\bullet$:*

$$\begin{tikzcd} W \arrow[r, tail, "w"] \arrow[d, shift right=2, "d_0"'] \arrow[d, shift left=2, "d_1"] & Y_1 \arrow[d, shift right=2, "d_0^{Y_\bullet}"'] \arrow[d, shift left=2, "d_1^{Y_\bullet}"] \\ U \arrow[r, tail, "u"'] \arrow[u, "s_0" description] & Y_0 \arrow[u, "s_0^{Y_\bullet}" description] \end{tikzcd}$$

is necessarily a groupoid.

3. The theorem

From now on, we shall suppose that $\mathbb{E}$ is a regular Mal'tsev category.

3.1. Lemma. *Let $\mathbb{E}$ be a regular Mal'tsev category. Then the category $Grd\mathbb{E}$ of internal groupoids in $\mathbb{E}$ is a fully faithful Birkhoff subcategory of the category $RGh\mathbb{E}$ of internal reflexive graphs in $\mathbb{E}$.*

Proof. As defined for instance in [17], a Birkhoff subcategory is a subcategory which is stable under subobjects, products and homomorphic images. The category $Grd\mathbb{E}$ is clearly stable under products in $RGh\mathbb{E}$, and stable under monomorphisms following the characterization given by Proposition 2.6. It remains to show it is stable under homomorphic images inside $RGh\mathbb{E}$ which is a regular category as soon as so is $\mathbb{E}$. The regular epimorphisms in $RGh\mathbb{E}$ are the levelwise regular epimorphisms in $\mathbb{E}$. So, consider the following diagram depicting a regular epimorphism in $RGh\mathbb{E}$ and suppose that the vertical left hand side reflexive graph is underlying a groupoid:

$$\begin{tikzcd}
X_1 \arrow[r, two heads, "f_1"] \arrow[d, shift right=3, "d_0^{X_\bullet}"'] \arrow[d, shift left=3, "d_1^{X_\bullet}"] & Y_1 \arrow[d, shift right=3, "d_0^{Y_\bullet}"'] \arrow[d, shift left=3, "d_1^{Y_\bullet}"] \\
X_0 \arrow[r, two heads, "f"'] \arrow[u, "s_0^{X_\bullet}" description] & Y_0 \arrow[u, "s_0^{Y_\bullet}" description]
\end{tikzcd}$$

First observe that, since the Mal'tsev category is regular, the downward square indexed by 0 has, by Proposition 2.5, a regular epic factorization towards the pullback of d_0 along f. Accordingly, the map f_1 produces a regular epic factorization $R(f_1) : R[d_0^{X_\bullet}] \twoheadrightarrow R[d_0^{Y_\bullet}]$.

Then complete the diagram by the horizontal kernel equivalence relations of the two lower levels:

$$\begin{tikzcd}
R_2[f] \arrow[r, equal] \arrow[d, shift right=2, "d_0"'] \arrow[d, shift left=2, "d_1"] \arrow[d, bend left=60, "d_2"] & R[d_0^{X_\bullet}] \arrow[r, shift left=2, "R(d_1^{f_1})"] \arrow[r, shift right=2, "R(d_0^{f_1})"'] \arrow[d, shift right=2, "d_0"'] \arrow[d, shift left=2, "d_1"] & R[d_0^{X_\bullet}] \arrow[l] \arrow[r, two heads, "R(f_1)"] \arrow[d, shift right=3, "d_0^{X_\bullet}"'] \arrow[d, "d_1^{X_\bullet}"] \arrow[d, shift left=3, "d_2^{X_\bullet}"] & R[d_0^{Y_\bullet}] \arrow[d, shift right=2, "d_0^{Y_\bullet}"'] \arrow[d, shift left=2, "d_1^{Y_\bullet}"] \arrow[d, dotted, bend left=60, "d_2"] \\
R_1[f] \arrow[r, equal] \arrow[d, shift right=3] \arrow[d, shift left=3] & R[f_1] \arrow[r, shift left=2, "d_1^{f_1}"] \arrow[r, shift right=2, "d_0^{f_1}"'] \arrow[d, shift right=3, "R(d_0^{X_\bullet})"'] \arrow[d, shift left=3, "R(d_1^{X_\bullet})"] & X_1 \arrow[l] \arrow[r, two heads, "f_1"] \arrow[d, shift right=3, "d_0^{X_\bullet}"'] \arrow[d, shift left=3, "d_1^{X_\bullet}"] & Y_1 \arrow[d, shift right=3, "d_0^{Y_\bullet}"'] \arrow[d, shift left=3, "d_1^{Y_\bullet}"] \\
R[f] \arrow[r, equal] \arrow[u] & R[f] \arrow[r, shift left=2, "d_1^f"] \arrow[r, shift right=2, "d_0^f"'] \arrow[u] & X_0 \arrow[l] \arrow[r, two heads, "f"'] \arrow[u, "s_0^{X_\bullet}" description] & Y_0 \arrow[u, "s_0^{Y_\bullet}" description]
\end{tikzcd}$$

So, we get the vertical left and side reflexive graph $R[f_1]$ which is a subreflexive graph of the internal groupoid $X_\bullet \times X_\bullet$. According to Proposition 2.6, it is actually underlying an internal groupoid $R_1[f]$ since $\mathbb{E}$ is a Mal'tsev category; whence the map d_2 which produces the groupoid structure on the left hand side. Now complete the diagram by the kernel relations of the vertical maps indexed by 0. Since the factorization $R(f_1)$ is a regular epimorphism, namely the quotient map of the upper middle horizontal kernel relation, it produces a factorization d_2 on the vertical right hand side which, in turn, makes the vertical right hand side reflexive graph Y_1 an internal groupoid according to Theorem 1.3. ∎

This is essentially the same idea as in Theorem 3.1 of [12], but, here, with a sharper formulation and a more structural proof. We shall suppose now that, in addition, the category $\mathbb{E}$ has pushouts of split monomorphisms along any regular epimorphism f :

$X \twoheadrightarrow Y$ or, equivalently, is such that the base-change functor $f^* : Pt_Y\mathbb{E} \to Pt_X\mathbb{E}$ along any map of this kind has a left adjoint, see [4].

3.2. THEOREM. *Let $\mathbb{E}$ be a regular Mal'tsev category having pushouts of split monomorphisms along any regular epimorphism. Given any internal groupoid $X_\bullet$ and any regular epimorphism $f : X_0 \twoheadrightarrow Y$, the reflexive graph determined by the pushout of $s_0^{X_\bullet}$ along f:*

$$\begin{tikzcd} X_1 \arrow[r, two heads, "f_1"] \arrow[d, shift right=2, "d_0^{X_\bullet}"'] \arrow[d, shift left=2, "d_1^{X_\bullet}"] & Y_1 \arrow[d, dotted, shift right=2, "d_0"'] \arrow[d, dotted, shift left=2, "d_1"] \\ X_0 \arrow[r, two heads, "f"'] \arrow[u, "s_0^{X_\bullet}" description] & Y \arrow[u, "s_0" description] \end{tikzcd}$$

is underlying an internal groupoid which, accordingly, is the codomain of the cocartesian internal functor above the regular epimorphism f with respect to the forgetful functor $(\ )_0 : Grd\mathbb{E} \to \mathbb{E}$.

PROOF. It is a straightforward consequence of the previous lemma once observed that f being a regular epimorphism, so is f_1 as the pushout of a regular epimorphism. ∎

This theorem has two meaningful corollaries:

3.3. COROLLARY. *Under the assumptions of the previous theorem, given any regular epimorphism $f : X \twoheadrightarrow Y$, the pushout of s_0^f along f produces a split epimorphism $(\psi_f, 0_f)$ underlying an internal abelian group in the slice category $\mathbb{E}/Y$:*

$$\begin{tikzcd} R[f] \arrow[r, two heads, "\chi_f"] \arrow[d, shift right=2, "d_0^f"'] \arrow[d, shift left=2, "d_1^f"] & \vec{A}_f \arrow[d, shift left=2, "\psi_f"] \\ X \arrow[r, two heads, "f"'] \arrow[u, "s_0^f" description] & Y \arrow[u, shift left=1, "0_f"] \end{tikzcd}$$

which makes the pair (f, χ_f) an internal functor in $\mathbb{E}$. The regular epimorphism f has an abelian kernel equivalence relation $R[f]$ as soon as the above downward square indexed by 0 is a pullback. When, in addition, $\mathbb{E}$ is exact, this condition becomes characteristic of the regular epimorphisms with abelian kernel equivalence relations. In particular, an object X is affine if and only if the following downward square is a pullback:

$$\begin{tikzcd} X \times X \arrow[r, two heads, "\chi_X"] \arrow[d, shift right=2, "p_0^X"'] \arrow[d, shift left=2, "p_1^X"] & \vec{A}_X \arrow[d, shift left=2, "\psi_X"] \\ X \arrow[r, two heads, "\tau_X"'] \arrow[u, "s_0^X" description] & 1 \arrow[u, shift left=1, "0"] \end{tikzcd}$$

PROOF. Since f coequalizes d_0^f and d_1^f, the groupoid structure on Y has his two legs d_0 and d_1 equal to ψ_f, namely it produces an internal (abelian) group on the object ψ_f in the slice category $\mathbb{E}/Y$, which implies that $R[\psi_f]$ is an abelian equivalence relation. So, when the square in question is a pullback, the map d_0^f has itself an abelian kernel equivalence relation which is equivalent to saying that $R[f]$ is an abelian equivalence relation.

When $\mathbb{E}$ is exact the converse is true thanks to the construction of the *direction* of an affine object with global support (see [6]) in the Mal'tsev slice category $\mathbb{E}/Y$ which produces the desired pullback. ∎

3.4. Corollary. *Under the assumptions of the last theorem, when (f, s) is a split epimorphism, the split epimorphism $(\psi_f, 0_f)$ is the universal abelian object associated with (f, s) in the pointed unital fiber $Pt_Y\mathbb{E}$:*

$$\begin{tikzcd}
X \arrow[r, tail, "s_1"] \arrow[d, shift right, "f"'] & R[f] \arrow[r, two heads, "\chi_f"] \arrow[d, shift right, "d_0^f"'] & \vec{A}_f \arrow[d, shift right, "\psi_f"'] \\
Y \arrow[r, tail, "s"] \arrow[u, shift right, "s"'] \arrow[rr, bend right, equal] & X \arrow[r, two heads, "f"] \arrow[u, shift right, "s_0^f"'] & Y \arrow[u, shift right, "0_f"']
\end{tikzcd}$$

with the above universal comparison map which is a regular epimorphism in $Pt_Y\mathbb{E}$, and thus makes $\chi_f.s_1$ a regular epimorphism in $\mathbb{E}$.

Proof. The fact that the pair (f, χ_f) is underlying an internal functor means the following identity $(*)$ which implies the three others:

$$\begin{array}{ll}
(*)\ \chi_f(a,b) + \chi_f(b,c) = \chi_f(a,c); & (1)\ -\chi_f(a,b) + \chi_f(a,c) = \chi_f(b,c) \\
(2)\ \chi_f(0,b) = -\chi_f(b,0); & (3)\ \chi_f(a,b) = \chi_f(a,0) - \chi_f(b,0)
\end{array}$$

Let us show that $\chi_f.s_1 : X \to \vec{A}_f$ is the universal comparison with the free abelian object in $Pt_Y\mathbb{E}$. So, let $(\alpha, \omega) : A \rightleftarrows Y$ be an abelian object in $Pt_Y\mathbb{E}$ and $g : X \to A$ be any map in $Pt_Y\mathbb{E}$ between (f, s) and (α, ω). Consider the following diagram where $\partial_\alpha : R[\alpha] \to A$ is the internal opsubtraction of the abelian group structure on α:

$$\begin{tikzcd}
 & R[\alpha] \arrow[rr, "\partial_\alpha"] & & A \\
R[f] \arrow[ur, "R(g)"] \arrow[rr, two heads, "\chi_f"] & & \vec{A}_f \arrow[ur, dotted, "\bar{g}"] & \\
X \arrow[u, "s_0^f"] \arrow[rr, two heads, "f"'] & & Y \arrow[u, "0_f"] \arrow[uur, "\omega"'] &
\end{tikzcd}$$

Then $\partial_\alpha(g(a), g(a)) = g(a) - g(a) = \omega f(a)$ and the universal property of $\vec{A}_f$ produces a map $\bar{g} : \vec{A}_f \to A$ such that $\bar{g}\chi_f(a,b) = \partial_\alpha(g(a), g(b)) = g(b) - g(a)$. Thus $\bar{g}\chi_f(0,a) = g(a)$ which means: $(**) : g = \bar{g}.(\chi_f.s_1)$. It remains to show the uniqueness of such a factorization satisfying $(**)$. Let $\phi : \vec{A}_f \to A$ be another group homomorphism such that $g = \phi.\chi_f.s_1$. Then:

$$\begin{aligned}
\phi(\chi_f(a,b)) &= \phi(\chi_f(0,b) - \chi_f(0,a)) = \phi(\chi_f(0,b)) - \phi(\chi_f(0,a)) \\
&= g(b) - g(a) = \bar{g}(\chi_f(a,b))
\end{aligned}$$

Now, since χ_f is a regular epimorphism, we get the desired uniqueness $\phi = \bar{g}$. Finally the universal comparison $\chi_f.s_1 : X \to \vec{A}_f$ is a regular epimorphism in $Pt_Y\mathbb{E}$, since, by the previous lemma, $Ab(Pt_Y\mathbb{E})$ appears as a Birkhoff subcategory of $Pt_Y\mathbb{E}$. ∎

Two final remarks about this last corollary.

1) From [2] (by Theorem 2.2.9, Corollary 1.8.20 and Proposition 1.7.5), we knew how to associate with any split epimorphism $(f, s) : X \rightleftarrows Y$ a universal abelian object in the fiber $Pt_Y\mathbb{E}$, but it was in a less straightforward way, namely via the following coequalizer in $Pt_Y\mathbb{E}$:

$$\begin{tikzcd}
X \arrow[r, tail, shift left, "{(1,sf)}"] \arrow[r, tail, shift right, "{(sf,1)}"'] \arrow[dr, shift right, "f"'] & R[f] \arrow[r, two heads, "q_f"] \arrow[d, shift right, "d_0^f.f"'] & \vec{A}_f \arrow[dl, shift right, "\psi_f"'] \\
& X \arrow[ul, shift right] \arrow[u, shift right, "s_0^f.s"'] \arrow[ur, shift right, "0_f"'] &
\end{tikzcd}$$

2) This corollary enlarges an observation made in the context of subtractive categories in [9]. First, in [14], Mal'tsev categories are also characterized in the following way:

3.5. Proposition. *A category $\mathbb{E}$ is a Mal'tsev one if and only if any fiber $Pt_Y\mathbb{E}$ is a subtractive category.*

This time, the notion of subtractive category is inspired by the notion of subtractive varieties [20]: a pointed category is said to be *subtractive* when, for any relation $(d_0, d_1) : R \rightarrowtail X \times X$ on X, if s_0^X and $(1_X, 0) : X \rightarrowtail X \times X$ factors through (d_0, d_1), then $(0, 1_X) : X \rightarrowtail X \times X$ factors through (d_0, d_1) as well. A major fact in this context is the following:

3.6. Proposition. [8] *In a subtractive category, on any object X, there is at most one abelian group structure.*

Later on, in [9], the abelianization functor for regular subtractive categories was described in the following way:

3.7. Proposition. *In a regular subtractive category $\mathbb{E}$, the abelianization of any object X, when it exists, is given by the cokernel of the diagonal $s_0^X : X \to X \times X$.*

Clearly, via Proposition 3.5, our last corollary appears as the natural extension of the previous proposition to the context of Mal'tsev categories.

References

[1] M. Barr, *Exact categories*, Springer L.N. in Math., **236** (1971), 1-120.

[2] F. Borceux and D. Bourn : *Mal'cev, Protomodular, Homological and Semi-Abelian Categories*, Kluwer, Mathematics and Its Applications, vol. 566, 2004, 479 pp.

[3] D. Bourn, *The shift functor and the comprehensive factorization for internal groupoids*, Cahiers Topologie Gom. Diffrentielle Catg. **28** (1987), 199-226.

[4] D. Bourn, *Normalization equivalence, kernel equivalence and affine categories*, Springer L.N. in Math., **1488** (1991), 43-62.

[5] D. Bourn, *Mal'cev Categories and fibration of pointed objects*, Applied categorical structures **4** (1996), 302-327.

[6] D. Bourn, *Baer sums and fibered aspects of Mal'cev operations*, Cahiers de Top. et Géom. Diff. **40** (1999), 297-316.

[7] D. Bourn and M. Gran, *Centrality and connectors in Maltsev categories*, Algebra univers. **48** (2002), 317-343.

[8] D. Bourn and Z. Janelidze, *Subtractive categories and extended subtractions*, Applied categorical structures **17** (2009), 302-327.

[9] D. Bourn and Z. Janelidze, *A note on the abelianization functor*, Communications in Algebra **44** (2016), 2009-2033.

[10] A. Carboni, J. Lambek and M.C. Pedicchio, *Diagram chasing in Mal'cev categories*, J. Pure Appl. Algebra, **69** (1991), 271-284.

[11] A. Carboni, M.C. Pedicchio and N. Pirovano, *Internal graphs and internal groupoids in Mal'cev categories*, CMS Conference Proceedings **13** (1992), 97-109.

[12] M. Gran, *Internal categories in Mal'cev categories*, J. Pure Appl. Algebra, **143** (1999), 221-229.

[13] G. Janelidze and M.C. Pedicchio, *Pseudogroupoids and commutators*, Theory and Applications of Categories **8** (2001), 408-456.

[14] Z. Janelidze, *Subtractive categories*, Applied categorical structures **13** (2005), 343-350.

[15] B. Jónsson and A. Tarski, *Direct decompositions of finite algebraic systems*, Notre-Dame Math. Lectures, Notre-Dame, Indiana (1947).

[16] A.I. Mal'tsev, *On the general theory of algebraic systems*, Matematicheskii Sbornik, N.S. **35** (1954), 3-20.

[17] E.G. Manes, *Algebraic theories*, Springer, Graduate Texts in Mahtematics, **26** (1976).

[18] M.C. Pedicchio, A categorical approach to commutator theory, Journal of Algebra **177** (1995), 647-657.

[19] J.D.H. Smith, *Mal'cev varieties*, Springer L.N. in Math. **554** (1976).

[20] A. Ursini, *On subtractive varieties*, Algebra univers. **31** (1994), 204-222.

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