Theory and Applications of Categories, Vol. 37, No. 5, 2021, pp. 107–143.

DEPENDENT PRODUCTS AND 1-INACCESSIBLE UNIVERSES

GIULIO LO MONACO

Abstract. The purpose of this writing is to explore the exact relationship running between geometric ∞-toposes and Mike Shulman's proposal for the notion of elementary ∞-topos, and in particular we will focus on the set-theoretical strength of Shulman's axioms, especially on the last one dealing with dependent sums and products, in the context of geometric ∞-toposes. Heuristically, we can think of a collection of morphisms which has a classifier and is closed under these operations as a well-behaved internal universe in the ∞-category under consideration. We will show that this intuition can in fact be made to a mathematically precise statement, by proving that, once fixed a Grothendieck universe, the existence of such internal universes in geometric ∞-toposes is equivalent to the existence of smaller Grothendieck universes inside the bigger one. Moreover, a perfectly analogous result can be shown if instead of geometric ∞-toposes our analysis relies on ordinary sheaf toposes, although with a slight change due to the impossibility of having true classifiers in the 1-dimensional setting. In conclusion, it will be shown that, under stronger assumptions positing the existence of intermediate-size Grothendieck universes, examples of elementary ∞-toposes with strong universes which are not geometric can be found.

Contents

The author was supported by the grant MUNI/A/1186/2018 of Masaryk University, Brno.
Received by the editors 2020-02-06 and, in final form, 2021-02-05.
Transmitted by Richard Garner. Published on 2021-02-10.
2020 Mathematics Subject Classification: 03E55, 18N60.
Key words and phrases: higher categories, higher toposes, elementary higher toposes, Grothendieck universes, large cardinals, dependent products, dependent sums, classifiers, generic morphisms.

1. Introduction

The present writing aims to find the precise relationship running between the notion of geometric, or Grothendieck ∞-topos, and that of elementary ∞-topos as proposed by Mike Shulman[1], in terms of the assumptions made on our set-theoretical foundation. In the 1-categorical case, an elementary 1-topos is a locally Cartesian closed category with finite limits and colimits and a subobject classifier, and it is widely known that every geometric (or Grothendieck) 1-topos is an elementary 1-topos. If one translates this notion of elementary topos naively to the ∞-categorical context, then one can still prove that every geometric ∞-topos is a naive elementary ∞-topos. However, since elementary ∞-toposes are supposed to be environments for modeling a semantics for homotopy type theory (see [VV.13] and [Shu19] respectively for what is meant here by homotopy type theory and modeling a semantics for it), we would like to add extra axioms positing the existence of internal universes closed under suitable operations. One possibility is to require that every family of objects live in a "weak" universe, closed under finite limits, finite colimits and dependent sums. A stronger possibility is to require that every family of objects live in a "strong" universe, which is also closed under dependent products. The main results of this paper show that every geometric ∞-topos is an elementary ∞-topos with weak universes (Theorem 3.7), while the statement that every geometric ∞-topos is an elementary ∞-topos with strong universes is equivalent to a large cardinal axiom (Theorem 3.13).

Work about the interaction between (non-homotopy) intuitionistic type theory and chains of universes, with mentions of the expectancy that these should have inaccessible cardinals as their set-theoretical counterpart, has already been done, for example, in [Pal91], [Pal98] and [RGP98], in which a study is undertaken from the syntactic point of view of what their authors call superuniverses, corresponding to what we here treat semantically and call 1-inaccessible universes. In turn, their super-superuniverses should correspond to our 2-inaccessible universes, and so on. Following this thread, [Rat93] proceeds with a proof-theoretic study of type theory with universes, while [Rat00] and [Rat01] do the same but taking type theory with superuniverses into account. The present work can be considered part of the same stream of investigations, in a somewhat complementary fashion, that is category-theoretic rather than proof-theoretic, semantic rather than syntactic. Also, we want to point out that in all the aforementioned works there is nothing homotopical or admitting a higher-category-theoretic interpretation. This should not be surprising since, while the idea of elementary ∞-topos is by its very conception necessarily homotopical, there is nothing intrinsically homotopical in each of its axioms and, in particular, in the notion of internal universes. One of the reasons that make us more interested in the ∞-categorical formulation is that all geometric ∞-toposes are known to have such internal universes (Theorem 2.9), whereas these almost always fail to exist in geometric 1-toposes, so that the universe axiom of elementary ∞-topos is never satisfied

[1] To the best of the author's knowledge, this definition was first foreshadowed in [Shu12] and later made to a more precise proposal in [Shu17]

by geometric toposes, when formulated in a 1-categorical setting. For this reason, we will resort to a weakening of the notion of classifier (Definition 2.14), that will allow us to carry on a parallel discussion of 1-toposes and ∞-toposes, while keeping the strong version of it solely for the homotopical case (Theorem 3.13). However, the stronger notion can be and has been investigated in some classes of 1-categories. The reader might look for example at [MP02], whose authors use their definition of stratified pseudotopos, which comprises a chain of universes, again with a connection with type theory in mind, but in a purely 1-categorical context.

This work will be thus structured: in section 2, the categorical tools that are needed later on will be given with appropriate references; section 3 is devoted to the definitions of elementary ∞-toposes, the statement of our main theorem, and an analysis of the extent to which the inclusion of geometric into elementary is always true; section 4 contains a proof that the indicated set-theoretical assumption entails in fact the desired inclusion; section 5 proves that it is minimal among such assumptions; in section 6, a precise account is given of the strength of isolated parts of the definition of elementary ∞-topos; finally, section 7 makes a stronger set-theoretical assumption and constructs by its means a family of examples of elementary ∞-toposes that are not geometric.

2. Preliminaries

We start by giving a brief account of how geometric ∞-toposes are defined and some references for some of the ∞-categorical tools that will be used, the theory lying behind them requiring far too much space to be fully presented here. The objects we are going to analyse are often just called ∞-toposes, but we prefer to use the adjective "geometric", firstly in order to remind ourselves that the notion we are handling only constitutes an analog of the 1-categorical geometric toposes, a.k.a. sheaf toposes, a.k.a. Grothendieck toposes, and secondly to better distinguish them from the other kind of objects that we will be faced with, which throughout this writing will be referred to as elementary ∞-toposes (with more specific denominations, as we will see) and whose definition is meant to be the ∞-categorical analog of what elementary toposes do in 1-category theory.
We assume familiarity with the theory of ∞-categories in the language of weak Kan complexes, and we refer the reader to [Lur09] for the details thereof. In particular, section 5.5 in [Lur09] deals with the theory of presentable ∞-categories. We only recall some of the results that we are going to use repeatedly. The vast majority of our statements applies to both the case of geometric toposes and geometric ∞-toposes. Whenever it so happens, we will bracket the symbol ∞ to signal that it may be taken into account or ignored at will, yielding analogous results in the two contexts. We also give specific references for the 1-categorical statements, although in many cases the same results will be attained by specializing the ∞-categorical ones to the case of nerves.
The statements of [Lur09], Corollary 5.3.4.15 and Remark 5.3.4.16, or [AR94], Proposition 1.16 and the subsequent remark, say:

2.1. Proposition. *For a regular cardinal κ, the collection of κ-compact objects in an (∞-)category is stable under κ-small colimits and retracts.*

Putting together [Lur09], Propositions 5.5.3.5, 5.5.3.10, 5.5.3.11 and 5.5.6.18, we also obtain the following nice stability properties:

2.2. Proposition. *Let $\mathcal{C}$ a presentable ∞-category. Then:*

- *For a simplicial set S, the ∞-category $\operatorname{Fun}(S, \mathcal{C})$ is presentable.*
- *For a functor $p : S \to \mathcal{C}$, the ∞-category $\mathcal{C}_{/p}$ is presentable.*
- *For a functor as above, the ∞-category $\mathcal{C}_{p/}$ is presentable.*
- *For an integer $n \geq -2$, the ∞-category $\tau_n\mathcal{C}$ of n-truncated objects of $\mathcal{C}$ is presentable.*

The 1-categorical version of the first statement can be found in [AR94], Proposition 1.54; the second and third are the content of [AR94], Proposition 1.57 (at least in the case we will use, but they can be adapted to the general case, see for example [AR94], exercise 2.h); the fourth reduces to trivial statements in 1-categories, except when $n = -1$; in this case, it gives the category of subobjects of the terminal object, which is a complete lattice and therefore presentable (see, for example, [AR94], Example 1.10(5)).
We call $\mathcal{S}$ the ∞-category of Kan complexes, or ∞-groupoids. Since this is equivalent to the ∞-category presented by the Quillen model structure on **Top**, we will freely interchange the words "space" and "Kan complex" for the purposes of this writing. Furthermore, we will use the same notation $\mathcal{P}(\mathcal{C})$ to refer to the category of presheaves of sets whenever $\mathcal{C}$ is an ordinary category, and the ∞-category of presheaves of spaces whenever $\mathcal{C}$ is an ∞-category. We believe that it will always be unambiguously clear from the context which interpretation should be chosen.
The following two statements, found as [Lur09], Corollary 5.4.1.5 and Proposition 5.3.4.13 respectively, will later on prove of paramount importance in order to give the procedure for our main result a starting kick.

2.3. Proposition. *Let X be a Kan complex, and κ an uncountable regular cardinal. Then X is κ-compact as an object of $\mathcal{S}$ if and only if it is essentially κ-small, i.e. there is a κ-small Kan complex X' and a homotopy equivalence $X' \to X$.*

2.4. Proposition. *Let $\mathcal{C}$ be a presentable ∞-category, S a small simplicial set and $f : S \to \mathcal{C}$ a functor. For a regular cardinal $\kappa > |S|$, if for every vertex $s \in S$ the object $f(s)$ is κ-compact, then f is κ-compact as an object of $\operatorname{Fun}(S, \mathcal{C})$.*

Finally, [Lur09], Corollary 5.5.2.9 (or [AR94], Proposition 1.66 and its dual, simpler adaptation) probably deserves to be called one of the most important results in the theory of (∞-)categories:

2.5. Theorem. [Adjoint Functor Theorem, presentable version] *Let $f : \mathcal{C} \to \mathcal{D}$ be a functor between presentable (∞-)categories. Then*

- *The functor f has a right adjoint if and only if it preserves small colimits.*
- *The functor f has a left adjoint if and only if it is accessible and it preserves small limits.*

In conclusion to this section, we mention a couple of useful results about the computation of limits and colimits in ∞-categories, found respectively as Proposition 1.2.13.8 and Corollary 5.1.2.3 in [Lur09]. A straightforward explicit calculation gives the former for 1-categories, [ML71], Theorem V.3.1 gives the latter.

2.6. Proposition. *Let $\mathcal{C}$ be an (∞-)category, and $C \in \mathcal{C}$ an object therein. Then the projection $\mathcal{C}_{/C} \to \mathcal{C}$ preserves colimits.*

2.7. Proposition. *Let $\mathcal{C}$ be an (∞-)category, K and S simplicial sets (small categories), and assume that $\mathcal{C}$ admits all K-indexed colimits. Consider a functor $f : K \to \mathrm{Fun}(S, \mathcal{C})$. Then an extension $f^{\triangleright} : K^{\triangleright} \to \mathrm{Fun}(S, \mathcal{C})$ is a colimit diagram if and only if for every vertex $s \in S$ the induced functor $K^{\triangleright} \to \mathcal{C}$ obtained by evaluating at s is a colimit diagram.*

We now recall the definition of ∞-topos. There are quite a few equivalent definitions, which we deem useful to write explicitly. Recall the following definition, appearing in [Lur09], Definition 6.1.6.1:

2.8. Definition. *Let $\mathcal{C}$ be an ∞-category with pullbacks, and let S be a class of morphisms in $\mathcal{C}$ that is stable under pullbacks. For an object $C \in \mathcal{C}$, let us denote by $\mathcal{C}^{(S)}_{/X} \subseteq \mathcal{C}_{/X}$ the subcategory spanned by elements of S as objects and equivalences as morphisms. We say that the class S has a classifier if the functor $X \mapsto \mathcal{C}^{(S)}_{/X}$ is representable.*

In particular, a classifier is a morphism $f : U' \to U$ such that there are natural equivalences $\mathrm{Map}(X, U) \to \mathcal{C}^{(S)}_{/X}$ for every object X, given by pulling back f.

2.9. Theorem. *Given an ∞-category $\mathcal{X}$, the following conditions are equivalent:*

1. *There is a small ∞-category $\mathcal{C}$ such that $\mathcal{X}$ is a left exact accessible localization of $\mathcal{P}(\mathcal{C})$.*
2. *The ∞-categorical Giraud's axioms are satisfied:*
    - *$\mathcal{X}$ is presentable.*
    - *Colimits in $\mathcal{X}$ are universal.*
    - *Coproduts in $\mathcal{X}$ are disjoint.*
    - *Every groupoid object in $\mathcal{X}$ is effective.*

3. *$\mathcal{X}$ is presentable with universal colimits, and the functor*

$$\mathcal{X}^{op} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$$

taking every object $x \in \mathcal{X}$ to the ∞-category $\mathcal{X}_{/x}$ preserves limits.

4. *$\mathcal{X}$ is presentable with universal colimits, and for arbitrarily large regular cardinals κ, the class of relatively κ-compact morphisms has a classifier.*

5. *$\mathcal{X}$ is presentable with universal colimits, and for every regular cardinal κ such that κ-compact objects are stable under pullbacks, then the class of relatively κ-compact morphisms has a classifier.*

The equivalence between (1), (2) and (3) is obtained by combining [Lur09], Theorems 6.1.0.6 and 6.1.3.9, that between these three and (4) is [Lur09], Theorem 6.1.6.8. The discussion immediately preceding this theorem yields that the suitable cardinals mentioned in the statements are those for which the corresponding compact objects are stable under the formation of pullback diagrams. Therefore in order to see the equivalence between (4) and (5) it will suffice to verify that not only do such cardinals exist, but that they can be arbitrarily large. This will be clear with point 2 of Example 4.7 later on.

2.10. Remark. *The statement of [Lur09], Theorem 6.1.6.8 uses the wording "sufficiently large regular cardinals", rather than "arbitrarily large regular cardinals". This is not the correct statement and, in fact, the proof in the book leads to the version stated above here.*

2.11. Definition. *An ∞-category $\mathcal{X}$ is a geometric ∞-topos whenever it satisfies the equivalent conditions of Theorem 2.9.*

Among the above equivalent definitions, the only one that is straightforwardly an analog of the corresponding 1-categorical notion is the first. In fact, we also have a definition in terms of the Giraud's axioms, although these are slightly different from their ∞-categorical counterpart. Points (3), (4) and (5) above are not satisfied by 1-toposes, where there is something similar to classifiers for relatively κ-compact morphisms, that however do not give bijections of sets $\mathrm{Map}(X, U) \cong \mathcal{X}_{/X}^{(S_\kappa)}$, because automorphisms of objects living over X are not taken into account. In Proposition 4.17, we will put a partial remedy to that in the case of 1-toposes. There are other definitions as well, but we decided to only give the two mentioned ones here, because we will make no use of, say, any sheaf conditions in the following.

2.12. Proposition. *Given a category $\mathcal{X}$, the following conditions are equivalent:*

1. *There is a small category $\mathcal{C}$ such that $\mathcal{X}$ is a left exact accessible localization of $\mathcal{P}(\mathcal{C})$.*

2. *$\mathcal{X}$ satisfies Giraud's axioms:*

   - *$\mathcal{X}$ is presentable.*

- *Colimits in $\mathcal{X}$ are universal.*
- *Coproducts in $\mathcal{X}$ are disjoint.*
- *Every equivalence relation in $\mathcal{X}$ is effective.*

This can be found in [Lur09], Proposition 6.1.0.6, which expresses Giraud's axioms in a form slightly different from the usual one, in order to parallel it to the ∞-categorical ones.

2.13. Definition. *A category $\mathcal{X}$ is a geometric topos if it satisfies the equivalent definitions of Proposition 2.12.*

Before concluding the section, we give one last definition, introducing a weaker version of the notion of classifier, motivated by the fact that, as we have observed above, the existence of classifiers in geometric 1-toposes is an unreasonable requirement.

2.14. Definition. *Let $\mathcal{C}$ be an ordinary category with pullbacks, and let S be a class of morphisms in $\mathcal{C}$ that is stable under pullbacks. We say that $f : U' \to U$ is a generic morphism for S if, for every morphism $g : X \to Y$, there is a map $Y \to U$ and a pullback square*

$$\begin{array}{ccc} X & \longrightarrow & U' \\ \big\downarrow{\scriptstyle g} & & \big\downarrow{\scriptstyle f} \\ Y & \longrightarrow & U. \end{array}$$

We note that obviously every classifier is in particular a generic morphism for its class.

3. Elementary ∞-toposes

Before we get to Shulman's proposed definition for elementary ∞-topos, we will give some weaker sets of axioms and see to what extent geometric ∞-toposes satisfies those.

3.1. Definition. *An ∞-category $\mathcal{E}$ is called a naive elementary ∞-topos if it satisfies the following axioms:*

(E1) *$\mathcal{E}$ is finitely complete and cocomplete.*

(E2) *$\mathcal{E}$ is locally Cartesian closed, i.e. for every object $X \in \mathcal{E}$ the overcategory $\mathcal{E}_{/X}$ is Cartesian closed.*

(E3) *There is a classifier for the class of all monomorphisms, i.e. there is an object Ω and a monomorphism $* \to \Omega$ such that for every object $X \in \mathcal{E}$ the map $\mathrm{Map}(X, \Omega) \to \mathrm{Sub}(X)$ given by pulling back is an equivalence.*

We call this "naive" because it is simply a transposition of the 1-categorical definition of elementary topos, but it doesn't take into account the motivation that is supposed to lie behind it. In particular, homotopy type theory deals consistently with universes, but there is no sign in the definition above of anything that may be interpreted as universes. Therefore, naive elementary ∞-toposes are expected not to have any practical usage insofar as one is concerned with modeling a semantics for homotopy type theory, as already said in the introduction.

3.2. Remark. *In a geometric ∞-topos there is a class of morphisms, called effective epimorphisms, which is stable under pullbacks and has the property that every morphism factors as an effective epimorphism followed by a monomorphism. Moreover, whenever a morphism is both a monomorphism and an effective epimorphism, then it is an equivalence. For more details about this, see [Lur09], chapter 6, or [LM18].*

3.3. Proposition. *Every geometric ∞-topos $\mathcal{X}$ is a naive elementary ∞-topos.*

Proof. (E1) follows by presentability. Given a morphism $f : Y \to X$, the functor taking products with f in $\mathcal{X}_{/X}$ is the composite of the base change f^*, which preserves colimits because they are universal, and a functor which postcomposes f, which also preserves them because it is left adjoint to f^*. Therefore by the adjoint functor theorem we obtain (E2).
For (E3), we want to show that the association $X \to \mathrm{Sub}(X)$ is a representable functor. Using the characterization of [Lur09], Proposition 5.5.2.2, this amounts to showing that it takes values in small spaces and it preserves limits. For the first claim, observe that $\mathrm{Sub}(X)$ is equivalent to the subcategory of $\mathcal{X}_{/X}$ spanned by monomorphisms, which is $\tau_{-1}(\mathcal{X}_{/X})$, which is presentable by Proposition 2.2, therefore $\mathrm{Sub}(X)$ is generated under colimits. Since it is a poset, this means that all objects in there are least upper bounds of subposets of it, therefore there's only a small amount of them.
For the other claim, we already know by Theorem 2.9 that the association $X \mapsto \mathcal{X}_{/X}$ preserves limits, so that if $X = \operatorname{colim} X_i$ there is an equivalence $\mathcal{X}_{/X} \simeq \lim \mathcal{X}_{/X_i}$ given respectively in the two directions by pulling back along the inclusions $X_i \to X$ and by taking colimits. We only need to show that this restricts to the subcategories spanned by monomorphisms or, equivalently, that monomorphisms are stable under these two operations. That they are stable under pullback is clear. For the other direction, suppose that $f : Y \to X$ is a colimit of monomorphisms $f_i : Y_i \to X_i$. Using Remark 3.2, factor f as an effective epimorphism f^e followed by a monomorphism f^m. By universality of colimits, there are double pullback diagrams

$$\begin{array}{ccc} Y_i & \longrightarrow & Y \\ \big\downarrow{\scriptstyle f_i^e} & & \big\downarrow{\scriptstyle f^e} \\ Z_i & \longrightarrow & Z \\ \big\downarrow{\scriptstyle f_i^m} & & \big\downarrow{\scriptstyle f^m} \\ X_i & \longrightarrow & X \end{array}$$

such that the horizontal arrows are inclusions in colimits and the vertical composites are f_i and f respectively. Since f_i^m and f_i are monomorphisms by assumption, then so is f_i^e, but then Remark 3.2 says that it is an equivalence, therefore f^e is a colimit of equivalences and so itself an equivalence. This implies that f is a monomorphism. ■

Before going on to stronger definitions, we need a preliminary notion:

3.4. Definition. *Given a morphism f in an ∞-category (or an ordinary category) $\mathcal{C}$ admitting pullbacks, and denoting with f^* a pullback functor, we call dependent sum a left adjoint of f^*, and dependent product a right adjoint of f^*, if they exist, and in that case we write*

$$\sum_f \dashv f^* \dashv \prod_f .$$

If $f : X \to *$ is a morphism toward a terminal object, we will also write $\sum_X$ and $\prod_X$ respectively.

3.5. Remark. *By the adjoint functor theorem and universality of colimits, every morphism in a geometric ∞-topos admits both a dependent sum and a dependent product. Moreover, the dependent sum along f can by expressed as postcomposition with f, as a consequence of the pullback property.*
Dependent products are more complicated, but they have an explicit description as well. In the specific case of a terminal morphism $X \to *$ in $\mathcal{S}$ (or in **Set***), the dependent product of an object Z over X is the space (set) of sections of the structure morphism. This expression can be generalized to some extent in terms of exponentials, assuming we already have a way to compute these. With a slight abuse of notation (which ceases to be such in $\mathcal{S}$ and in* **Set***), we denote by $\{p\} \to X^Z$ a morphism from a terminal object which is adjunct to $p : Z \to X$. Now choose an object $p : Z \to X$ in $\mathcal{C}_{/X}$ and an object $W \in \mathcal{C}$, which pulls back to the object $pr_2 : W \times X \to X$ in $\mathcal{C}_{/X}$. Then*

$$\begin{aligned} \mathrm{Map}_X(W \times X, Z) &\simeq \mathrm{Map}(W \times X, Z) \times_{\mathrm{Map}(W\times X,X)} \{pr_2\} \\ &\simeq \mathrm{Map}(W, Z^X) \times_{\mathrm{Map}(W,X^X)} \{\tilde{pr_2}\} \\ &\simeq \mathrm{Map}(W, Z^X \times_{X^X} \{\mathrm{id}\}) \end{aligned}$$

which means that $\prod_X Z = Z^X \times_{X^X} \{\mathrm{id}\}$.
An analogous expression holds for generic dependent products, where we need to replace exponentials in $\mathcal{C}$ with exponentials as taken in overcategories of the form $\mathcal{C}_{/X}$.

3.6. Definition. *An ∞-category $\mathcal{E}$ is called an elementary ∞-topos with weak universes if it satisfies the axioms (E1) through (E3) of a naive elementary ∞-topos plus the following:*

- *(E4w) For every morphism $f \in \mathcal{E}$, there is a class S of morphisms such that $f \in S$, S has a classifier and it is closed under finite limits and colimits as taken in overcategories and under depentend sums.*

3.7. Theorem. *Every geometric ∞-topos $\mathcal{X}$ is an elementary ∞-topos with weak universes.*

The proof of this will be deferred until the next section, after we have introduced the technique of uniformization.
The following definition is the one originally proposed by Mike Shulman.

3.8. Definition. *An ∞-category $\mathcal{E}$ is called an elementary ∞-topos with strong universes if it satisfies the axioms (E1) through (E3) of a naive elementary ∞-topos plus the following:*

(E4) For every morphism $f \in \mathcal{E}$, there is a class S of morphisms such that $f \in S$, S has a classifier and it is closed under finite limits and colimits as taken in overcategories and under dependent sums and products.

As already stated in the introduction, the presence of the additional axiom, new to anyone only acquainted with ordinary elementary toposes, should become clear upon observing that, as elementary ∞-toposes should be exactly those ∞-categories providing a semantics for homotopy type theory, they should include an internal notion of universe closed under certain operations.
We now focus on the axiom (E4), and even more specifically, on a subaxiom thereof, reducing to a minimal non-trivial statement, which in fact will turn out to be equivalent to a large cardinal assumption. Before getting into any set-theoretical definitions, we just say what this subaxiom is.

3.9. Definition. *Let $\mathcal{C}$ be an ∞-category (or an ordinary category) with pullbacks and admitting dependent sums and products. We say that $\mathcal{C}$ satisfies the axiom (DepProd) if every morphism $f \in \mathcal{C}$ is contained in a class of morphisms S which has a generic morphism (Definition 2.14) and is closed under dependent products.*

Time has come to present the large cardinals we will need to deal with in the following.

3.10. Definition. *Let κ be a cardinal. We say that κ is 0-inaccessible if it is just inaccessible. Inductively, for an ordinal α we say that κ is α-inaccessible if it is inaccessible and for every ordinal $\beta < \alpha$ and every cardinal $\lambda < \kappa$ there exists a β-inaccessible cardinal μ such that $\lambda \leq \mu < \kappa$.*

3.11. Remark. *The above definition simply says that the set of β-inaccessibles smaller than κ is unbounded. Equivalently, since κ is regular, it says that the set of β-inaccessibles smaller than κ has cardinality κ.*

3.12. Remark. *In the following, we will mostly be interested in what it means for a Grothendieck universe $\mathcal{U}$ to be 1-inaccessible, i.e. the set of $\mathcal{U}$-small inaccessible cardinals is unbounded. If one wishes to work with the formalism of NBG set theory instead of TG set theory, this can be phrased by saying that inaccessible cardinals form a proper class.*

Now we are ready to state the main result which is the punchline of this writing and will be proven in the next sections.

3.13. THEOREM. *Choose a Grothendieck universe $\mathcal{U}$. The following conditions are equivalent:*

1. *$\mathcal{U}$ is 1-inaccessible.*
2. *Every geometric topos satisfies (DepProd).*
3. *Every geometric ∞-topos satisfies (DepProd).*
4. *Every geometric ∞-topos is an elementary ∞-topos with strong universes.*

3.14. REMARK. *Notice that the equivalence between (3) and (4) in the above theorem means in particular that the existence of the relevant generic morphisms in geometric ∞-toposes allows us to assume that they are in fact classifiers, as the proof will make even clearer. This is not the case for geometric toposes, as we already noted just before Proposition 2.12.*

As we have seen in Theorem 3.7, if we simply leave out dependent products from the formulation of the axiom (E4), each geometric ∞-topos already satisfies it. Moreover, as we will see in the next section, the strategy for proving (E4) is the same as that for proving (E4w), except that we need to also take care of dependent products in the process, which is exactly why we need 1-inaccessibility. Just before proving Theorem 3.13, we therefore give the much simplified proof of Theorem 3.7 in order to make clear what this strategy is.
We will first prove the implications (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3), for which the strategy is exactly the same. Then we will only prove the implication (3) $\Rightarrow$ (1), for which it suffices to just consider the ∞-category of Kan complexes, and see that (2) $\Rightarrow$ (1) follows by specializing to discrete Kan complexes, which can be identified with sets. Lastly, the proof (1) $\Rightarrow$ (4) is essentially the same as for (1) $\Rightarrow$ (3), and (4) $\Rightarrow$ (3) is trivial, as it will be reminded at the end of the next section.

4. 1-inaccessibility implies (DepProd)

One result of paramount importance for all that comes next is the so-called uniformization theorem. We learned it from [AR94] for the case of ordinary categories, and it is readily adapted to the context of ∞-categories.

4.1. LEMMA. *Let I be a poset, and let $\hat{I}$ a collection of subposets of I containing all finite sequences $\{i_0 \leq \ldots \leq i_n\}$ of elements of I, and let us order $\hat{I}$ by inclusion. Let then $\mathcal{C}$ be an ∞-category admitting colimits indexed by I and by elements of $\hat{I}$, and $g : \mathrm{N}(I) \to \mathcal{C}$ be a functor. Then there exists a functor $\hat{g} : \mathrm{N}(\hat{I}) \to \mathcal{C}$ which admits a colimit and is such that for every $M \in \hat{I}$ we have $\hat{g}(M) \simeq \mathrm{colim}_{i \in M}\, g(i)$ and $\mathrm{colim}\, \hat{g} \simeq \mathrm{colim}\, g$.*

Proof. Throughout this proof, we will freely refer to notations and statements drawn from [Lur09], section 4.2.3. Let us define a poset L as follows: the elements of L are the union of the elements of I and those of $\hat{I}$; for $i, i' \in I$, we say $i \preceq i'$ if and only if $i \leq i'$ in I, for $M, M' \in \hat{I}$ we say $M \preceq M'$ if and only if $M \subseteq M'$ in $\hat{I}$, plus we say that $i \preceq M$ if and only if $i \in M$. Note that both I and $\hat{I}$ can be regarded as subposets of L. Also, note that $\mathrm{N}(L)$ is a special case of the K_F mentioned in Notation 4.2.3.1, with $K = \mathrm{N}(I)$, $\mathcal{J} = \hat{I}$ and F selecting subposets.
So we can use Proposition 4.2.3.4 and conclude that g can be extended to a functor $g' : \mathrm{N}(L) \to \mathcal{C}$ in such a way that $g'(M) \simeq \operatorname{colim}_{i \in M} g(i)$ for every $M \in \hat{I} \subseteq L$, and moreover $\operatorname{colim} g \simeq \operatorname{colim} g'$. We want to show that $\hat{g} = g'_{|\mathrm{N}(\hat{I})}$ has the required property.
We want to use Corollary 4.2.3.10. The only part left to check is that the hypotheses of Proposition 4.2.3.8 are satisfied. In the notation used there, σ reduces to a sequence $\{i_0 \leq \ldots \leq i_n\}$, $\mathcal{J}_\sigma$ is the subposet of $\hat{I}$ of all M's containing this sequence, and this is contractible because it has an initial object, namely the sequence itself. Moreover, if σ is degenerate, i.e. repeats at least one of its terms, then $\mathcal{J}_\sigma = \mathcal{J}'_\sigma$, so we also have the other hypothesis. We conclude by using Corollary 4.2.3.10. ∎

4.2. Proposition. *Given two regular cardinals $\lambda < \mu$, the three following conditions are equivalent:*

1. *Every λ-accessible category $\mathcal{C}$ is also μ-accessible.*
2. *Every λ-accessible ∞-category $\mathcal{D}$ is also μ-accessible.*
3. *In each λ-filtered poset every μ-small subposet is contained in a μ-small λ-filtered subposet.*

Proof. The equivalence between (1) and (3) is given in [AR94], Theorem 2.11. That (2) implies (1) can be seen just by taking nerves and observing that accessibility of a category is precisely detected on its nerve. It remains to prove that (3) implies (2). Choose a λ-accessible ∞-category $\mathcal{D}$. Since every μ-filtered diagram is in particular λ-filtered, $\mathcal{D}$ admits in particular all small μ-filtered colimits. It remains to prove that it is generated under μ-filtered colimits by its μ-compact objects. Using [Lur09], Proposition 5.3.1.16 we know that every object $D \in \mathcal{D}$ is a colimit of a diagram $g : \mathrm{N}(I) \to \mathcal{C}$ of λ-compact objects, where I is a λ-filtered poset. Now define $\hat{I}$ to be the poset of all μ-small λ-directed subposets of I. This is μ-filtered since, given a μ-small collection $(M_j)_{j \in J}$ of such subposets, their union is also μ-small by regularity of μ, therefore by assumption (3) it is contained in a μ-small λ-filtered subposet of I, which is then an upper bound for the collection $(M_i)_{i \in I}$.
Now observe that I, $\hat{I}$ and $\mathcal{C}$ satisfy the hypotheses of Lemma 4.1, so we can obtain a functor $\hat{g} : \mathrm{N}(\hat{I}) \to \mathcal{D}$ having D as a colimit. Moreover, for every $M \in \hat{I}$, $\hat{g}(M) \simeq \operatorname{colim}_{i \in M} g(i)$, so it is a μ-small colimit of λ-compact (therefore μ-compact) objects, which is μ-compact by Proposition 2.1. This concludes the proof. ∎

4.3. Definition. *Given two regular cardinals $\lambda < \kappa$, we say that λ is sharply smaller than κ, and we write $\lambda \lhd \kappa$, if the equivalent conditions of Proposition 4.2 are verified.*

We list now a few nice properties of the sharply smaller relation.

4.4. Lemma.

1. *Given regular cardinals $\lambda < \mu$ such that for all cardinals $\alpha < \lambda$ and $\beta < \mu$ we have $\beta^\alpha < \mu$, then $\lambda \lhd \mu$.*
2. *Given arbitrary regular cadinals $\lambda \leq \mu$, then $\lambda \lhd (2^\mu)^+$.*
3. *Given a set of regular cardinals $(\lambda_i)_{i\in I}$, then there is a regular carinal μ such that for every $i \in I$ we have $\lambda_i \lhd \mu$.*
4. *Given two regular cardinals $\lambda < \mu$ such that μ is inaccessible, then $\lambda \lhd \mu$.*

Proof. (1) and (2) are respectively Example 2.13(4) and 2.13(3) in [AR94]. Taking $\mu' = \sup_{i\in I} \lambda_i$, we can apply (2) and thus obtain (3) by setting $\mu = (2^{\mu'})^+$. Finally, to prove (4), observe that the definition of inaccessible readily implies (1), hence the result. ■

4.5. Lemma. *Let $\lambda \lhd \kappa$ be regular cardinals, and let $\mathcal{C}$ be a λ-accessible ∞-category. Then an object $C \in \mathcal{C}$ is κ-compact if and only if it is a retract of a κ-small λ-filtered colimit of λ-compact objects.*

Proof. We already know one implication by Proposition 2.1, without even using the hypothesis of λ-filteredness. Conversely, assume that C is κ-compact. Since $\mathcal{C}$ is λ-accessible, we can express C as a colimit of λ-compact objects indexed by the nerve of a λ-filtered poset I. Let $\hat{I}$ be the poset of all λ-filtered κ-small subposets of I. Given less than κ objects in $\hat{I}$, then their union is still κ-small, therefore condition (3) in Proposition 4.2 yields that it is contained in an object of $\hat{I}$. This means that the poset $\hat{I}$ is κ-filtered. For each $M \in \hat{I}$, let B_M be a colimit for the diagram indexed by $\mathrm{N}(M)$. Using Lemma 4.1, we obtain a functor $\mathrm{N}(\hat{I}) \to \mathcal{C}$ sending each $M \in \hat{I}$ to B_M and having C as a colimit. Since C is κ-compact, id_C factors as $C \to B_M \to C$ for some M, which means that C is a retract of B_M and has therefore the desired description. ■

The next theorem is one of the main ingredients to prove the first direction of the implications (1) ⇒ (2) and (1) ⇒ (3) of Theorem 3.13. It is found as Theorem 2.19 in [AR94] in the case of ordinary categories. The proof presented here is entirely analogous to the original one.

4.6. Theorem. [Generalized Uniformization Theorem] *Given a small set of accessible functors $F_i : \mathcal{C}_i \to \mathcal{D}_i$ between presentable ∞-categories, there is a regular cardinal κ such that each F_i preserves κ-compact objects. Moreover, this remains true for every other cardinal $\kappa' \rhd \kappa$.*

Proof. By Lemma 4.4, there is a regular cardinal λ such that all involved ∞-categories are λ-accessible and all functors F_i's are λ-accessible. Consider all λ-compact objects of all ∞-categories $\mathcal{C}_i$'s. Since they form a small set, there is a regular cardinal $\mu \geq \lambda$ such that all of their images along the respective F_i's are μ-compact. Hence, using again Lemma 4.4 we can find a regular cardinal $\kappa \rhd \lambda$, so that in particular all images of λ-compact objects along the F_i's are κ-compact. We now show that κ-compact objects are preserved by each F_i.
Consider such an object $C \in \mathcal{C}_i$. By Lemma 4.5 we can write it as a retract of a κ-small λ-filtered colimit of λ-compact objects of $\mathcal{C}_i$. Since F_i is λ-accessible, such a colimit is preserved, and retractions are obviously preserved as well, thus $F_i(C)$ is a retract of a κ-small colimit of objects of $\mathcal{D}_i$, all of which are μ-compact by choice of μ. Therefore $F_i(C)$ is κ-compact.
Finally, if we choose a cardinal $\kappa' \rhd \kappa$, it is also sharply greater than λ, hence the same proof applies. ∎

The following example should shed some light on why uniformization provides such a useful technique when dealing with properties of compact objects.

4.7. Example.

1. *Let $\mathcal{C}$ be an accessible (∞-)category and let K be a small (∞-)category. Consider the set of all projections* $\mathrm{Fun}(K, \mathcal{C}) \to \mathcal{C}$ *given by evaluating on objects of K. Combining Proposition 2.4 and Theorem 4.6, we can find a regular cardinal κ such that κ-compact objects in* $\mathrm{Fun}(K, \mathcal{C})$ *are exactly those functors which take values in κ-compact objects of $\mathcal{C}$.*

2. *With the same notation as above, if we instead consider the limit functor* $\lim : \mathrm{Fun}(K, \mathcal{C}) \to \mathcal{C}$ *we get a cardinal κ such that κ-compact objects are stable under K-indexed limits. In particular, if we take K to be the cospan category, we obtain the implication* (5) ⇒ (4) *in Theorem 2.9.*

3. *Assume we have a small set of cardinals $(\kappa_i)_{i \in I}$ such that the set of κ_i-compact objects enjoys the property P_i, and also assume that all the properties P_i's are obtained through uniformization. Using Lemma 4.4 and Theorem 4.6 we find a cardinal κ such that κ-compact object enjoy each of the properties P_i's at the same time.*

4. *Given any of the preceding cases, suppose we can find a cardinal $\kappa' > \kappa$ such that κ' is inaccessible. Then the same result will also hold after replacing κ with κ', as Lemma 4.4 ensures. In particular, if the universe $\mathcal{U}$ is 1-inaccessible to start with, we can always assume that a cardinal found as in the examples above is inaccessible.*

First, we will put this technique to use by finally proving Theorem 3.7. The strategy for it will essentially make use of characterization (5) in Theorem 2.9 (which, as said, is also true in geometric 1-toposes, even though it doesn't characterize them). We need a prelimiary lemma.

4.8. Lemma. *Let $\mathcal{C}$ be a presentable (∞-)category with universal colimits, $Y \in \mathcal{C}$ an object and λ a regular cardinal. Then an object $p : X \to Y$ in the overcategory $\mathcal{C}_{/Y}$ is λ-compact if and only if X is λ-compact in $\mathcal{C}$.*

Proof. Assume $p : X \to Y$ is λ-compact in the overcategory, and take Z to be a λ-filtered colimit of objects $Z_j \in \mathcal{C}$. Then we write

$$\begin{aligned} \mathrm{Map}(X, \operatorname{colim} Z_j) &\simeq \mathrm{Map}_Y(X, Y \times \operatorname{colim} Z_j) && \text{by definition} \\ &\simeq \mathrm{Map}_Y(X, \operatorname{colim} Y \times Z_j) && \text{by universality of colimits and Prop. 2.6} \\ &\simeq \operatorname{colim} \mathrm{Map}_Y(X, Y \times Z_j) && \text{by } \lambda\text{-compactness} \\ &\simeq \operatorname{colim} \mathrm{Map}(X, Z_j) && \text{by definition.} \end{aligned}$$

Conversely, assume that X is λ-compact in $\mathcal{C}$. Consider a λ-filtered diagram of objects $w_j : W_j \to Y$ in the overcategory. For each one of them, the mapping space $\mathrm{Map}_Y(X, W_j)$ is an equalizer of the diagram

$$\mathrm{Map}(X, W_j) \underset{const_p}{\overset{w_j \circ -}{\rightrightarrows}} \mathrm{Map}(X, Y).$$

Since in **Set** (and in $\mathcal{S}$) λ-filtered colimits commute with λ-small limits, this implies the claim. ■

Proof Proof of Theorem 3.7. In view of Proposition 3.3, it suffices to prove (E4w). For a morphism $f : X \to Y$, the pullback functor $f^* : \mathcal{X}_{/Y} \to \mathcal{X}_{/X}$ is accessible by universality of colimits. By uniformization, we may choose a regular cardinal κ such that κ-compact objects are stable under pullback and f^* preserves κ-compact objects. In particular, Lemma 4.8 says that f is contained in the class S_κ of relatively κ-compact morphisms. We have to prove that S_κ is the desired class. By Theorem 2.9(5), S_κ has a classifier. For closure under dependent sums, remember Remark 3.5 and consider a double pullback diagram

$$\begin{array}{ccccc} X' & \longrightarrow & Y' & \longrightarrow & Z' \\ \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \end{array}$$

where $f, g \in S_\kappa$. This means that if Z' is κ-compact, then so is Y', and so is X'. By pasting law, we obtained that $g \circ f \in S_\kappa$.

For limits, it suffices to show the claim for a terminal object and for pullbacks. A terminal object in $\mathcal{X}_{/Y}$ is simply the identity on Y, which is clearly relatively κ-compact. Pullbacks in $\mathcal{X}_{/Y}$ are computed as pullbacks in $\mathcal{X}$. Given such a diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Z' \arrow[d] \arrow[ddr] & \\ Z'' \arrow[r] \arrow[drr] & Z \arrow[dr] & \\ & & Y \end{tikzcd}$$

where the structure maps of Z, Z' and Z'' are in S_κ, we need to prove that the structure map $X \to Y$ is also in S_κ. Choosing a κ-compact object U and a map $U \to X$, we know by assumption that the pullbacks of U along Z, Z' and Z'' is κ-compact. Now its pullback along X is seen to be a pullback of these three objects, which is κ-compact because κ-compact objects are assumed to be stable under pullback.
For colimits, choose a colimit $Z = \operatorname{colim} Z_i$ of a finite diagram in $\mathcal{X}_{/X}$, which is computed in $\mathcal{X}$ by Proposition 2.6. Choosing a κ-compact object U and a map $U \to X$, construct double pullback diagrams

$$\begin{tikzcd} V_i \arrow[r] \arrow[d] & V \arrow[r] \arrow[d] & U \arrow[d] \\ Z_i \arrow[r] & Z \arrow[r] & X. \end{tikzcd}$$

Now $Z_i \to X$ is relatively κ-compact for each i, then V_i is κ-compact. But V is now a finite colimit of κ-compact objects, therefore it is itself κ-compact, and the map $Z \to X$ is relatively κ-compact, as we desired. ∎

Our strategy to the proof of (1) ⇒ (2) and (1) ⇒ (3) in Theorem 3.13 is the same. The biggest problem will be closure under dependent products. Since they can be obtained in terms of plain exponentials (see Remark 3.5 again), we start tackling the issue by just looking at the case of exponentials of objects.

4.9. Proposition. *In the category* **Set** *of sets and functions, if $X, Y \in$* **Set** *are κ-compact for an inaccessible cardinal κ, then the exponential X^Y is κ-compact.*

Proof. Since κ-compact sets are precisely all those sets whose cardinality is smaller than κ, the conclusion is immediate in view of the inaccessibility of κ. ∎

4.10. Proposition. *In the ∞-category $\mathcal{S}$ of spaces, if $X, Y \in \mathcal{S}$ are κ-compact for an inaccessible cardinal κ, then the exponential X^Y is κ-compact.*

Proof. In view of Proposition 2.3 we know that, for all uncountable cardinals, being κ-compact is equivalent to being κ-small in the case of spaces. Therefore it suffices to prove that X^Y has less than κ cells, or equivalently less than κ n-cells in each dimension n. The set of n-cells is $\operatorname{Map}(Y \times \Delta^n, X)$. Now, since κ is inaccessible, for any two cardinals $\mu, \lambda < \kappa$ we have $\mu^\lambda < \kappa$, therefore if both spaces are uncountable we obtain set-theoretically the result. If one of them is not, it follows a fortiori. ∎

For the next proof, we will need a convenient expression for natural transformations between two functors. As one can find in [ML71], section IX.5 for the case of ordinary categories, or in [GHN15], section 5 for the case of ∞-categories, the set (space) of natural transformations between two functors F, G starting from a small (∞-)category $\mathcal{C}$ can be expressed as the end

$$\int_{C \in \mathcal{C}} \mathrm{Map}(F(C), G(C)).$$

Moreover, in both cases the end may be computed as a limit for a diagram indexed by a small (∞-)category whose cardinality is bounded by something which depends on $\mathcal{C}$.

4.11. Proposition. *Assume that the universe $\mathcal{U}$ is 1-inaccessible, and let $\mathcal{C}$ be a $\mathcal{U}$-small (∞-)category. Then there are arbitrarily large inaccessible cardinals κ such that κ-compact objects of $\mathcal{P}(\mathcal{C})$ are stable under exponentiation.*

Proof. Recalling Proposition 2.4, we may choose a cardinal $\mu > |\mathcal{C}|$ and therefore assume that every presheaf taking values in μ-compact sets (spaces) is μ-compact as an object of $\mathcal{P}(\mathcal{C})$. A usage of all four points in Example 4.7 will provide a cardinal κ such that:

1. All representable presheaves in $\mathcal{P}(\mathcal{C})$ are κ-compact;
2. κ-compact objects are stable under binary products;
3. Presheaves on $\mathcal{C}$ are κ-compact precisely when they take values in κ-compact sets (spaces);
4. κ is inaccessible;
5. Ends of diagrams in **Set** (in $\mathcal{S}$) indexed by objects of $\mathcal{C}$ and taking values in κ-compact objects are themselves κ-compact.

Now consider two κ-compact preshesaves $F, G \in \mathcal{P}(\mathcal{C})$, and consider their exponential F^G. By (3), it will suffice to prove that for every object $C \in \mathcal{C}$, the set (space) $F^G(C)$ is κ-compact. In view of the Yoneda lemma and the definition of exponential, we can compute

$$\begin{aligned} F^G(C) &\simeq \mathrm{Map}(\mathbf{y}(C), F^G) \\ &\simeq \mathrm{Map}(\mathbf{y}(C) \times G, F) \\ &\simeq \int_{D \in \mathcal{C}} \mathrm{Map}(\mathrm{Map}(D, C) \times G(D), F(D)). \end{aligned}$$

Now, by (1) and (3), $\mathrm{Map}(D, C)$ is κ-compact for every $D \in \mathcal{C}$, $F(D)$ and $G(D)$ are as well again by (3), therefore $\mathrm{Map}(D, C) \times G(D)$ also is by (2). Using then (4) and Proposition 4.9 (Proposition 4.10), we know that the term inside the end is κ-compact for every $D \in \mathcal{C}$, which immediately means that the whole end is κ-compact by applying (5), and the proof is complete. Moreover, this process can be repeated for arbitrarily large inaccessible cardinals, whose existence is granted by the hypothesis that the universe $\mathcal{U}$ is 1-inaccessible. ■

4.12. Lemma. *Let* $\mathcal{Y} \underset{i}{\overset{L}{\rightleftarrows}} \mathcal{X}$ *be a left exact localization between Cartesian closed presentable (∞-)categories. Then the functor i preserves exponentials.*

Proof. Since the Yoneda embedding of $\mathcal{Y}$ is fully faithful, it suffices to show that there is a natural equivalence between $\mathrm{Map}(A, i(Y^Z))$ and $\mathrm{Map}(A, i(Y)^{i(Z)})$ for every object $A \in \mathcal{Y}$. This follows from the following chain of natural isomorphisms (equivalences)

$$\begin{aligned}
\mathrm{Map}(A, i(Y^Z)) &\simeq \mathrm{Map}(L(A), Y^Z) && \text{by adjunction}\\
&\simeq \mathrm{Map}(L(A) \times Z, Y) && \text{by exponential property}\\
&\simeq \mathrm{Map}(L(A \times i(Z)), Y) && \text{by left exactness and fully-faithfulness}\\
&\simeq \mathrm{Map}(A \times i(Z), i(Y)) && \text{by adjunction}\\
&\simeq \mathrm{Map}(A, i(Y)^{i(Z)}) && \text{by exponential property}
\end{aligned}$$

■

4.13. Proposition. *Assume that the universe $\mathcal{U}$ is 1-inaccessible, and let $\mathcal{X}$ be a geometric (∞-)topos. Then there are arbitrarily large inaccessible cardinals κ such that κ-compact objects are stable under exponentiation.*

Proof. Since $\mathcal{X}$ is a geometric (∞-)topos, there is a small (∞-)category $\mathcal{C}$ and a left exact localization of the form

$$\mathcal{P}(\mathcal{C}) \underset{i}{\overset{L}{\rightleftarrows}} \mathcal{X}.$$

The statement is already true in $\mathcal{P}(\mathcal{C})$ since it is exactly Proposition 4.11. Enlarging κ if necessary by further uniformizing, we may additionally assume that i is κ-accessible and preserves κ-compact objects. Now consider two κ-compact objects $X, Y \in \mathcal{X}$. We wish to show that their exponential X^Y is still κ-compact. To this end, consider a κ-filtered colimit of objects Z_j's in $\mathcal{X}$. We have the following chain of isomorphisms (equivalences):

$$\begin{aligned}
\mathrm{Map}(X^Y, \operatorname{colim} Z_j) &\simeq \mathrm{Map}(i(X^Y), i(\operatorname{colim} Z_j)) && \text{by fully-faithfulness}\\
&\simeq \mathrm{Map}(i(X)^{i(Y)}, \operatorname{colim} i(Z_j)) && \text{by Lemma 4.12 and assumption on } i\\
&\simeq \operatorname{colim} \mathrm{Map}(i(X)^{i(Y)}, i(Z_j)) && \text{by Prop. 4.11 and assumption on } i\\
&\simeq \operatorname{colim} \mathrm{Map}(X^Y, Z_j) && \text{by Lemma 4.12 and fully-faithfulness}
\end{aligned}$$

■

Now that the case of exponentials is settled, we may move forward and start considering more general dependent products. We need a few lemmas first. Given two objects $X \to Y$ and $Z \to Y$ in an overcategory $\mathcal{X}_{/Y}$, we will denote their exponential therein as $(Z^X)_{/Y}$

4.14. Lemma. *Let $\mathcal{X}$ be a geometric (∞-)topos and $Y \in \mathcal{X}$ an object, and take two objects $X \to Y, Z \to Y \in \mathcal{X}_{/Y}$. Then for a morphism $f : Y' \to Y$ in $\mathcal{X}$ the pullback of $(Z^X)_{/Y}$ along f is equivalent to $(Z'^{X'})_{/Y'}$, where $X', Z' \in \mathcal{X}_{/Y'}$ are the respective pullbacks of X and Z along f.*

Proof. Since the Yoneda embedding of $\mathcal{X}_{/Y'}$ is fully faithful, it suffices to show that for every object W over Y' the mapping spaces $\mathrm{Map}_{Y'}(W, f^*(Z^X)_{/Y})$ and $\mathrm{Map}_{Y'}(W, (Z'^{X'})_{/Y'})$ are naturally equivalent. To show this, observe first that calculating binary products in $\mathcal{X}_{/Y}$ is the same as calculating pullbacks over Y in $\mathcal{X}$, and therefore pasting law applied to the double pullback diagram

$$\begin{array}{ccccc} \bullet & \longrightarrow & f^*X & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ W & \longrightarrow & Y' & \xrightarrow{f} & Y \end{array}$$

says that the bullet corner is equivalently occupied by $X \times \sum_f W$ of $\sum_f(f^*X \times W)$ (products computed in the overcategory). Moreover, the isomorphism (equivalence) between these objects is natural, since arising from the combined universal properties of $\sum_f$, binary products and f^*. Now we have a chain of natural isomorphisms (equivalences)

$$\begin{aligned}
\mathrm{Map}_{Y'}(W, f^*(Z^X)_{/Y}) &\simeq \mathrm{Map}_Y(\sum_f W, (Z^X)_{/Y}) && \text{by dependent sum property} \\
&\simeq \mathrm{Map}_Y(X \times \sum_f W, Z) && \text{by exponential property} \\
&\simeq \mathrm{Map}_Y(\sum_f (f^*X \times W), Z) && \text{by the observation above} \\
&\simeq \mathrm{Map}_{Y'}(f^*X \times W, f^*Z) && \text{by dependent sum property} \\
&= \mathrm{Map}_{Y'}(X' \times W, Z') && \text{by definition of } X' \text{ and } Z' \\
&\simeq \mathrm{Map}_{Y'}(W, (Z'^{X'})_{/Y'}) && \text{by exponential property}
\end{aligned}$$

■

4.15. Lemma. *Assume that the universe $\mathcal{U}$ is 1-inaccessible, and let $\mathcal{X}$ a geometric (∞-)topos. Then there are arbitrarily large inaccessible cardinals κ such that, uniformly for all κ-compact objects $Y \in \mathcal{X}$, κ-compact objects in the overcategories $\mathcal{X}_{/Y}$ are stable under exponentiation.*

Proof. *Step 1.* Suppose $\mathcal{X}$ is λ-accessible. By Proposition 4.13, we know that for every λ-compact object Y there is a cardinal that stabilizes the respective compact objects in $\mathcal{X}_{/Y}$ under exponentiation. Since λ-compact objects of $\mathcal{X}$ form a small set, by 1-inaccessibility we can take κ to be inaccessible and bigger than all these cardinals, thus uniformizing simultaneously in all the overcategories $\mathcal{X}_{/Y}$ with Y being λ-compact. Therefore we obtain the statement in the special case where Y is λ-compact. We have to show that it can be extended to all κ-compact objects.
Step 2. Uniformizing once more if necessary, we may further assume that κ-compact objects are stable under pullbacks in $\mathcal{X}$. Now pick a κ-compact object $Y \in \mathcal{X}$. Since $\mathcal{X}$ is λ-accessible, by Lemma 4.5 Y can be expressed as a retract of a κ-small λ-filtered colimit of λ-compact objects Y_j's. We denote the retraction at issue by $r : Y' \to Y$. Take two κ-compact objects $X \to Y, Z \to Y$ of $\mathcal{X}_{/Y}$ and call $X' = r^*X$ and $Z' = r^*Z$. By Lemma 4.8 $X, Z \in \mathcal{X}$ are κ-compact, and by stability under pullbacks X' and Z' are as well, therefore for every inclusion $a_j : Y_j \to Y'$ in the colimit, $X_j = a_j^*X'$ and $Z_j = a_j^*Z'$ are κ-compact.
Now, by Lemma 4.14 we have a double pullback diagram

$$\begin{array}{ccccc}
(Z_j^{X_j})_{/Y_j} & \longrightarrow & (Z'^{X'})_{/Y'} & \longrightarrow & (Z^X)_{/Y} \\
\downarrow & & \downarrow & & \downarrow \\
Y_j & \xrightarrow{\quad a_j \quad} & Y' & \xrightarrow{\quad r \quad} & Y
\end{array}$$

which, by universality of colimits in $\mathcal{X}$, exhibits $(Z'^{X'})_{/Y'}$ as a colimit of $(Z_j^{X_j})_{/Y_j}$'s, which are all κ-compact by step 1. Since $(Z^X)_{/Y}$ is now seen to be a retract of $(Z'^{X'})_{/Y'}$ which is a κ-small colimit of κ-compact objects, Lemma 4.5 says that $(Z^X)_{/Y}$ is itself κ-compact as an object of $\mathcal{X}$, therefore by Lemma 4.8 also as an object of $\mathcal{X}_{/Y}$, which is exactly what we wanted. ∎

4.16. Theorem. *Assume that the universe $\mathcal{U}$ is 1-inaccessible. Then in every geomeric (∞-)topos $\mathcal{X}$ there are arbitrarily large inaccessible cardinals κ such that the class S_κ of relatively κ-compact morphisms is stable under taking dependent products, i.e. the dependent product of a morphism in S_κ along another morphism in S_κ is itself in S_κ.*

Proof. Choose a cardinal κ (necessarily inaccessible by the proofs of the preceding propositions) such that κ-compact objects are stable under pullbacks and the statement of Lemma 4.15 holds. We now show that this is already the cardinal we are looking for.
Consider two relatively κ-compact morphisms $p : Z \to X$ and $f : X \to Y$. We want to show that the dependent product $\prod_f Z \to Y$ of p along f is relatively κ-compact, therefore consider a κ-compact object Y' and a morphism $g : Y' \to Y$. We have to prove that $g^* \prod_f Z$ is κ-compact.
Form a double pullback diagram

$$\begin{array}{ccccc} Z' & \longrightarrow & X' & \xrightarrow{f'} & Y' \\ \downarrow & & \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ Z & \xrightarrow{p} & X & \xrightarrow{f} & Y. \end{array}$$

We claim that $g^* \prod_f Z$ is equivalent to $\prod_{f'} Z'$ in the overcategory $\mathcal{X}_{/Y'}$. By Yoneda, it is enough to prove it after composing with $\mathrm{Map}_{Y'}(A, -)$ for any other object A over Y'. Observe that, by pasting law, $f^* \sum_g A$ is isomorphic (equivalent) to $\sum_{g'} f'^* A$, and it is so in a natural way, because the relevant isomorphism (equivalence) arises from the combined universal properties of the two relevant dependent sums and pullbacks. We now have a chain of natural isomorphisms (equivalences):

$$\begin{aligned} \mathrm{Map}_{Y'}(A, g^* \prod_f Z) &\simeq \mathrm{Map}_Y(\sum_g A, \prod_f Z) && \text{by dependent sum property} \\ &\simeq \mathrm{Map}_X(f^* \sum_g A, Z) && \text{by dependent product property} \\ &\simeq \mathrm{Map}_X(\sum_{g'} f'^* A, Z) && \text{by the observation above} \\ &\simeq \mathrm{Map}_{X'}(f'^* A, g'^* Z) && \text{by dependent sum property} \\ &\simeq \mathrm{Map}_{Y'}(A, \prod_{f'} Z') && \text{by dependent product property} \end{aligned}$$

which proves the claim true.
Since both f and p are relatively κ-compact and Y' is κ-compact, then so are X' and Z'. We finish the proof recalling that by Remark 3.5 $\prod_{f'} Z'$ is computed as $(Z'^{X'})_{/Y'} \times_{(X'^{X'})_{/Y'}} Y'$, which is κ-compact in view of Lemma 4.15 and stability under pullbacks. ∎

So far, we have proceeded in parallel treating the cases of 1-toposes and ∞-toposes likewise. For our last step, we would like to use Theorem 2.9(5), but as we know, there can be no true classifiers in the 1-dimensional case (see the discussion preceding Proposition 2.12). Therefore, we need to obtain a separate statement, giving at least enough generic morphisms. The proof of the following statement will be deferred to an appendix.

4.17. Proposition. *In a geometric topos, there are arbitrarily large regular cardinals κ such that the class S_κ of relatively κ-compact morphisms has a generic morphism.*

4.18. Corollary. [1] ⇒ (2), (3] *Assume that the universe $\mathcal{U}$ is 1-inaccessible. Then for every geometric (∞-)topos $\mathcal{X}$ and every morphism $f \in \mathcal{X}$ there is a class of morphisms $S \ni f$ such that S has a generic morphism and is closed under dependent products.*

Proof. Pick a morphism $f : X \to Y$ in $\mathcal{X}$. By universality of colimits, the induced functor $f^* : \mathcal{X}_{/Y} \to \mathcal{X}_{/X}$ is accessible, therefore we may apply Theorem 4.6 to see that it preserves κ-compact objects for some κ. Using Lemma 4.8, this means precisely that f is relatively κ-compact. The class S_κ of relatively κ-compact morphisms contains f and, by Example 4.7 and Proposition 4.17 (or Theorem 2.9(5)), we may further uniformize and therefore assume that S_κ has a generic morphism (or even a classifier, in a geometric ∞-topos) and that the statement of Theorem 4.16 holds true, which completes the proof. ■

4.19. Remark. *By following the steps leading to the proof of Corollary 4.18, the rest of axiom (E4) in Definition 3.8 is automatically satisfied. Namely, if κ-compact objects are stable under pullback, S_κ is always closed under dependent sums and the relevant finite limits and colimits, because we can retrace the proof of Theorem 3.7. Therefore, keeping in mind the brief discussion right after Definition 3.8, the same proof also shows the implication* (1) $\Rightarrow$ (4) *in Theorem 3.13. Finally, the implication* (4) $\Rightarrow$ (3) *is trivial, since (DepProd) is a subaxiom of (E4).*

4.20. Remark. *Going back to the definition of geometric ∞-topos, in the flavor of Theorem 2.9(4), one might think of the condition on relatively κ-compact morphisms as "there are enough internal universes". On the same wake, the axiom (E4) should be interpreted as a strengthening thereof, namely "there are enough internal universes that are closed under dependent products". The implication* (3) $\Rightarrow$ (4) *in Theorem 3.13 means that, when working in ∞-toposes, assuming (DepProd) already implies the existence of the latter kind of universes.*

5. (DepProd) implies 1-inaccessibility

In this section, we will establish both converse implications of those proven in the previous section. We first focus on the statement (3) $\Rightarrow$ (1) of Theorem 3.13, which is slightly more complicated than proving (2) $\Rightarrow$ (1), and then observe how our proof of the former already includes in itself a proof of the latter. As the following theorem shows, we may actually start from the apparently weaker assumptions that (DepProd) only hold in the category of sets, or in the ∞-category of spaces.

5.1. Lemma. *Let κ be an uncountable cardinal. Then κ is inaccessible if and only if the following condition holds:*

- *Given an arbitrary set I such that $|I| < \kappa$ and a family of cardinals $(\alpha_i)_{i\in I}$ such that for all $i \in I, \alpha_i < \kappa$, then we have*

$$\prod_{i\in I} \alpha_i < \kappa.$$

Proof. Suppose that the condition above is satisfied. To prove that κ is regular, take I and α_i's as in the hypothesis, and observe that, if $\alpha_i \geq 2$ for all i's, which of course we can safely assume, then

$$\sum_{i\in I} \alpha_i \leq \prod_{i\in I} \alpha_i$$

(this is essentially a consequence of Cantor's inequality $2^\lambda > \lambda$; for a proof, see [Jec06], formula 5.17, right before Lemma 5.9). Then regularity follows a fortiori by hypothesis. Given a cardinal $\lambda < \kappa$, we may choose a set I of cardinality λ and $\alpha_i = 2$. Then the hypothesis says precisely that

$$2^\lambda = \prod_{i \in I} 2 < \kappa$$

so that κ is also a strong limit.
Conversely, assume that κ is inaccessible. Given I and α_i's as in the hypothesis, choose sets X_i's of cardinalities α_i respectively. We need to show that $\prod_{i \in I} X_i$ has cardinality strictly smaller than κ. Now this can be described as the set of sections of the canonical map $\coprod_{i \in I} X_i \to I$, which is a subset of the exponential $(\coprod_{i \in I} X_i)^I$. In turn, identifying each function with its graph, we see that this is a subset of the power set $\mathcal{P}(I \times \coprod_{i \in I} X_i)$. By the assumption that κ is a strong limit, it suffices to show that $I \times \coprod_{i \in I} X_i$ is strictly smaller than κ. Now $\coprod_{i \in I} X_i$ is strictly smaller than κ by regularity, therefore the result follows by applying regularity once more, and observing that $I \times Y = \coprod_{i \in I} \{i\} \times Y$ for an arbitrary set Y. ∎

5.2. THEOREM. [3) $\Rightarrow$ (1] *Fix a universe $\mathcal{U}$. Suppose that for every morphism f in the ∞-category $\mathcal{S}$ of $\mathcal{U}$-small spaces there is a class S of morphisms containing f such that S has a generic morphism and it is closed under dependent products. Then the universe $\mathcal{U}$ is 1-inaccessible.*

PROOF. We must show that, given an arbitrary cardinal μ, there exists an inaccessible cardinal κ such that $\kappa > \mu$. Pick a cardinal μ and consider a discrete space X of cardinality μ. Then choose a suitable class S as in the hypothesis containing the terminal morphism $X \to \Delta^0$. It will have a generic morphism $t : \bar{U} \to U$. In particular, every fiber of t is an element of S, and X is such a fiber. Now consider the full subspace of $U_0 \subseteq U$ spanned by all vertices whose respective fiber is discrete, therefore equivalent to the nerve of a set, and take a pullback diagram

$$\begin{array}{ccc} \bar{U}_0 & \longrightarrow & \bar{U} \\ \downarrow{\scriptstyle t_0} & & \downarrow{\scriptstyle t} \\ U_0 & \longrightarrow & U. \end{array}$$

For every map of the form $Y \to \Delta^0$ which is contained in S and such that Y is a set, any classifying map factors through U_0 by construction, therefore t_0 classifies in particular all terminal morphisms in S whose domain is a set.
Before pointing to the cardinal we need, let us observe a nice property of t_0. Pick a fiber Z of t_0 and, for each $z \in Z$, choose a point of U_0. This will define a map $Z \to U_0$ which, in turn, yields a map $Y \to Z$. Now, calling Y_z the fiber of z, we obtain double pullback diagrams of the form

$$\begin{array}{ccccc} Y_z & \longrightarrow & Y & \longrightarrow & \bar{U}_0 \\ \downarrow & & \downarrow & & \downarrow \\ \Delta_0 & \xrightarrow{z} & Z & \longrightarrow & U_0 \end{array}$$

which, by universality of colimits in $\mathcal{S}$, means that the map $Y \to Z$ is equivalent to the projection $p : \coprod_{z \in Z} Y_z \to Z$. Now this projection belongs to S by construction, and the terminal map $Z \to \Delta_0$ also belongs to S by assumption, therefore the terminal map from the dependent product $\prod_Z p$ belongs to S. But by Remark 3.5 this can be expressed as the space of sections of p, which in turn may be identified with $\prod_{z \in Z} Y_z$. Since products of discrete spaces are themselves discrete, we obtain the following property:

- Given a set Z which is a fiber of t_0 and a set $(Y_z)_{z \in Z}$ of fibers of t_0 indexed by Z, then the product $\prod_{z \in Z} Y_z$ is also a fiber of t_0.

Now consider the set E of fibers of t_0, and denote the fiber of a vertex x as F_x. Since E is a small set, we can define a cardinal $\kappa = \sup_{x \in E} |F_x|$. Now we have that $\mu < \kappa$, because $|X| < |X|^{|X|} = |X^X|$, and X^X is certainly contained in E by the property $\bullet$, since it can be expressed as the product $\prod_{x \in X} X$. Similarly, for any set $F_x \in E$ it is true that $|F_x| < \kappa$. The proof will be complete by showing that κ is inaccessible.
We may safely assume uncountability just by picking μ infinite in the first place. In order to complete the proof, we will use the characterization given in Lemma 5.1. To this end, consider a family $(\alpha_i)_{i \in I}$ such that $|I| < \kappa$ and $\alpha_i < \kappa$ for each $i \in I$. By definition of κ, there are sets $F_I, F_{\alpha_i} \in E$ such that $|F_I| \geq |I|, |F_{\alpha_i}| \geq \alpha_i$. So there is an injective function $I \to F_I$ that, by adding copies of X if necessary, allows us to index the sets F_{α_i}'s over F_I. Hence we obtain

$$\prod_{i \in I} \alpha_i \leq \prod_{i \in F_I} |F_{\alpha_i}| = |\prod_{i \in F_I} F_{\alpha_i}| < \kappa$$

since, using the property $\bullet$ again, we know that the set $\prod_{i \in F_I} F_{\alpha_i}$ belongs to E, which is what we wanted. ■

5.3. Remark. [2) ⇒ (1] *The proof of Theorem 5.2 proceeds in an entirely analogous way on plain sets, the only exception being the first part when we reduce the generic morphism t to a less generic morphism t_0 only working for sets. Leaving out this first step, the proof also applies in* **Set***, hence becoming precisely the statement* (2) ⇒ (1) *in Theorem 3.13.*

6. Products are stronger than sums

The combination of both direction of (1) ⇔ (2) as well as (1) ⇔ (3) in Theorem 3.13 says that dependent products are a somewhat stronger notion than dependent sums. This does not mean that closure of a class of morphisms under the former already implies closure under the latter, but that the existence of nice classes of morphisms which are closed under dependent products implies that they can always be extended to classes which are closed under both dependent products and sums. Moreover, in the ∞-dimensional case we can always assume that generic morphisms are in fact classifiers.

6.1. Definition. *Let $\mathcal{C}$ be a (∞-)category with pullbacks and admitting dependent sums and products. We say that $\mathcal{C}$ satisfies the axiom* (*Dep*) *if every morphism $f \in \mathcal{C}$ is contained in a class of morphisms S which has a generic morphism and is closed under dependent products and sums.*

6.2. Corollary. *Assume that* (*DepProd*) *holds in the ∞-category of spaces (or in the category of sets). Then the apparently stronger axiom* (*Dep*) *already holds in every geometric (∞-)topos. Moreover, an arbitrary class of morphisms S in a geometric (∞-)topos, which has a generic morphism and is closed under dependent products can be extended to a class of morphisms S' which has a generic morphism and is closed under both dependent products and sums.*

Proof. Let us choose a geometric (∞-)topos $\mathcal{X}$. Since (*DepProd*) holds, an arbitrary morphism f is contained in a class which has a generic morphism and is closed under the formation of dependent products. Choose a class S of morphisms as above, and let $t : \bar{U} \to U$ be its generic morphism. By universality of colimits, the induced base change functor $t^* : \mathcal{X}_{/U} \to \mathcal{X}_{/\bar{U}}$ is accessible, therefore an application of Theorem 4.6 will tell us that it preserves κ-compact objects for some κ. Using Lemma 4.8, this means that t is relatively κ-compact. Since every morphism of S is a pullback of t, this means that $S \subseteq S_\kappa$, where S_κ is the class of relatively κ-compact morphisms. Now, an application of Corollary 4.18 yields that the universe we're working in is 1-inaccessible, therefore we may retrace our steps in the proof of Theorem 4.16, enlarging κ if necessary and thus ensuring that S_κ have a classifier and be closed under dependent products. Moreover, it is also clearly closed under dependent sums, since they are just compositions. The proof is then complete by taking $S' = S_\kappa$. ∎

6.3. Corollary. *Assume that* (*DepProd*) *holds either in the ∞-category of spaces or in the category of sets. Then, in a geometric ∞-topos (but* not *in a geometric* 1*-topos) the classes S' as in the corollary above can be assumed to satisfy a stronger version of* (*Dep*)*, where the phrase "generic morphism" is replaced by "classifier".*

Proof. Retracing the proof of the above corollary, we see that, by Theorem 2.9, the classes S_κ may always be taken to have a classifier. ∎

Corollary 6.2 shows that under the assumption that there is a generic morphism, then closure under dependent products almost implies closure under dependent sums as well. We want to conclude this section by showing that a generic morphism alone, on the other hand, a priori has nothing to do with closure under either operation. In order to accomplish this, we will exhibit a class of morphisms which has a generic morphism but is closed under neither dependent sums nor dependent products.

6.4. Example. *Let us work in* **Set**. *Let κ be a cardinal which is not regular, and define S as the class of all functions between sets such that all of their fibers are strictly smaller than κ. We claim that this class admits a generic morphism.*
Let's consider κ as a set. For each cardinal $\lambda < \kappa$, consider the map of sets $\lambda \to \kappa$ which is constant at the element of κ corresponding to λ. This yields a map

$$t : \coprod_{\lambda<\kappa} \lambda \to \kappa$$

which belongs to S since, by construction, the fibers are simply all λ's indexing the coproduct in the domain. Now consider a function $f : A \to B$ with all fibers being strictly smaller than κ. We may rewrite it as the natural map $\coprod_{b\in B} A_b \to B$, where the A_b's are the fibers of f. We now define a map $q : B \to \kappa$ in such a way that f will be a pullback of t along q.
For an element $b \in B$, set $q(b)$ to be the λ such that $|A_b| = \lambda$. An easy check will reveal that there is a pullback square

$$\begin{array}{ccc} \coprod_{b\in B} A_b & \longrightarrow & \coprod_{\lambda<\kappa} \lambda \\ \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle t} \\ B & \xrightarrow{\quad q \quad} & \kappa \end{array}$$

where the upper horizontal map chooses a total ordering for every set A_b. Moreover, the map q is even unique with this property. Indeed, assume there is another map q' having the same property. Since $q' \neq q$, there exists $b \in B$ such that $q'(b) = \lambda' \neq |A_b|$. Now we can build a double pullback diagram

$$\begin{array}{ccccc} A_b & \longrightarrow & \coprod_{b\in B} A_b & \longrightarrow & \coprod_{\lambda<\kappa} \lambda \\ \big\downarrow & & \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle t} \\ \{b\} & \longrightarrow & B & \xrightarrow{\quad q' \quad} & \kappa \end{array}$$

and, by pasting law, we see that A_b is isomorphic to the fiber of t corresponding to λ', which is absurd, since such a fiber of t should have cardinality λ'.
Now, observe that the class S just constructed is not closed under dependent sums. For example, consider a family of sets $(X_i)_{i\in I}$ such that $|I| < \kappa$ and $|X_i| < \kappa$ but $|\coprod_{i\in I} X_i| \geq \kappa$ (this is possible since κ has been chosen not to be regular). The canonical map $p : \coprod_{i\in I} X_i \to I$ belongs to S and so does the terminal map $I \to *$, but their composite does not, which is immediate by the construction of these maps.
To show that S is not closed under dependent products either, let us take the same family $(X_i)_{i\in I}$, assuming without loss of generality that none of these sets are empty or singletons. We use the formula given in Remark 3.5 to obtain that the dependent product of the projection p along the terminal map $I \to *$ is the (terminal map of the) set of sections of p, that is, $\prod_{i\in I} X_i$. This is bigger than $\coprod_{i\in I} X_i$ (see Lemma 5.1), therefore a fortiori it doesn't belong to S.

6.5. Example. *Example 6.4 shows that a generic morphism does not force dependent sums or products not even if it is a particularly convenient one, namely granting uniqueness of the classifying maps. In fact, not even a true classifier has anything to do with dependent sums and products. Indeed, working in the ∞-category $\mathcal{S}$, we can give the ∞-categorical analog of Example 6.4. Consider the projection map $\tilde{\mathcal{S}}^{\kappa}_{*/} \to \tilde{\mathcal{S}}^{\kappa}$, where the codomain is the core of the ∞-category of κ-small spaces. This map is even a classifier for the class of all maps having κ-small fibers, but for the same reason as above, this class is not closed under dependent sums or products.*

As a concluding observation, we want to note that, if under the assumption of there being a generic morphism we can establish that dependent products play a somewhat stronger role than dependent sums, a generic morphism itself does not partake in this hierarchy.
Indeed, the class of all morphisms in a geometric (∞-)topos is obviously closed under dependent products and sums, but it does not admit a generic morphism, so closure under either of them does not imply its existence. Conversely, Example 6.4 and Example 6.5 already show that classifiability implies neither closure. Moreover, even classifiability and dependent sums together do not imply dependent products, as it is easily seen by considering the class of relatively κ-compact morphisms in a geometric (∞-)topos without the 1-inaccessibility assumption.

7. Geometric $\subsetneq$ elementary

We have shown in Section 4 that, under the assumption that the chosen Grothendieck universe is 1-inaccessible, every geometric ∞-topos is also an elementary ∞-topos with strong universes. This gives an inclusion of the class of geometric ∞-toposes into the class of elementary ∞-toposes with strong universes. We now might wonder whether this inclusion is strict. In other words, whether there exist elementary ∞-toposes with strong universes that are not geometric. A partial answer is given in the following, where we provide a sufficient condition for the existence of such objects. Where hitherto we used the condition of 1-inaccessibility on the Grothendieck universe itself, now we will be using it on cardinals *within* this universe, and conclude that assuming that there are enough such cardinals there exists a plethora of examples of the kind of objects we are looking for.
First, as a technical step, we present a statement which is already known for ordinary categories and certainly expected for ∞-categories. Namely, if a small category is either complete or cocomplete, then it is a preorder, in the sense that every homset has at most one element. The technicalities involved here are not exceedingly long or complicated, but nonetheless there's a good chance that a complete depiction of them at this stage would drive the reader's thoughts away from the main point that we want to focus on right now. For this reason, we defer the proof of the Lemma 7.1 to an appendix, trusting that it will anyway seem like a reasonable result to expect.

7.1. Lemma. *Suppose $\mathcal{C}$ is an essentially small ∞-category that has either all small limits or all small colimits. Then for any choice of objects $X, Y \in \mathcal{C}$ the mapping space* $\mathrm{Map}(X, Y)$ *is either empty or contractible.*

7.2. Lemma. *A geometric ∞-topos cannot be small, unless it is equivalent to a point.*

Proof. Suppose there is a small ∞-topos $\mathcal{X}$. By Lemma 7.1, all of its mapping spaces are either empty or contractible. In particular, since for any two objects, $A, B \in \mathcal{X}$ we have $\mathrm{Map}(A, B) \simeq \mathrm{Map}(A, B) \times \mathrm{Map}(A, B)$, then the binary coproduct of two copies of the same object A is A itself. Now, by Theorem 2.9 coproducts are disjoint, therefore there is a pullback square

$$\begin{array}{ccc} \emptyset & \longrightarrow & A \\ \downarrow & & \downarrow \\ A & \longrightarrow & A. \end{array}$$

Obviously, replacing $\emptyset$ with A will also give a pullback square, therefore $A \simeq \emptyset$, so every object is initial, and $\mathcal{X}$ is equivalent to a point. ∎

We can now finally prove the theorems that we wanted. The author does not know whether or not the following statements admit a converse, or whether there are elementary ∞-toposes with strong universes which do not arise as in the cases exhibited below.

7.3. Proposition. *Fix a Grothendieck universe $\mathcal{U}$. Assuming that there is a $\mathcal{U}$-small 1-inaccessible cardinal κ, then the ∞-topos of spaces admits a subcategory which is an elementary ∞-topos with strong universes but not a geometric ∞-topos.*

Proof. Consider the full subcategory $\mathcal{S}^\kappa \subseteq \mathcal{S}$ spanned by all κ-compact spaces. We will show that this is already the object that we are looking for. Observe that since κ is inaccessible, there is a Grothendieck universe $\mathcal{U}_\kappa \in \mathcal{U}$ associated to it, and that $\mathcal{S}^\kappa$ is precisely the ∞-topos of spaces in the universe $\mathcal{U}_\kappa$. Theorem 3.13 now says exactly that $\mathcal{S}^\kappa$ is an elementary ∞-topos with strong universes in $\mathcal{U}_\kappa$, but since the definition of elementary ∞-topos with strong universes does not address any size issues, $\mathcal{S}^\kappa$ will also be such an object in $\mathcal{U}$.
We show now that it is not a geometric ∞-topos. Indeed, if it were, Lemma 7.2 would imply that it is a point, which is obviously not the case. ∎

7.4. Proposition. *Fix a Grothendieck universe $\mathcal{U}$. Given a small ∞-category $\mathcal{C}$, assume that there exists a 1-inaccessible cardinal $\kappa > |\mathcal{C}|$. Then there is a subcategory of $\mathcal{P}(\mathcal{C})$ that is an elementary ∞-topos with strong universes but not a geometric ∞-topos.*

Proof. As in Proposition 7.3, consider the full subcategory $\mathcal{P}(\mathcal{C})^\kappa \subseteq \mathcal{P}(\mathcal{C})$ spanned by all κ-compact objects. Again, since κ is inaccessible, there is a Grothendieck universe $\mathcal{U}_\kappa \in \mathcal{U}$ associated to it. Moreover, since $\kappa > |\mathcal{C}|$, Proposition 2.4 says that $\mathcal{P}(\mathcal{C})^\kappa$ is a presheaf category in the smaller universe, and therefore a geometric ∞-topos $\mathcal{U}_\kappa$. As above, this implies that it is an elementary ∞-topos with strong universes in $\mathcal{U}$.
To see that $\mathcal{P}(\mathcal{C})^\kappa$ cannot be a geometric ∞-topos, use Lemma 7.2 again. ∎

7.5. Remark. *For the next proposition, we will need the notion of 2-inaccessibility. In TG set theory, for a cardinal to be 2-inaccessible just means that the set of 1-inaccessibles below it is unbounded. In the NBG axiomatic system, the assumption of a 2-inaccessible universe translates to saying that there is a proper class of 1-inaccessible cardinals, whilst the notion of 1-inaccessibility goes back to our original definition.*

7.6. Proposition. *Fix a* 2*-inaccessible Grothendieck universe* $\mathcal{U}$*. Then every geometric* ∞*-topos* $\mathcal{X}$ *admits a subcategory which is an elementary* ∞*-topos with strong universes but not a geometric* ∞*-topos.*

Proof. Consider a diagram

$$\mathcal{X} \underset{i}{\overset{L}{\leftrightarrows}} \mathcal{P}(\mathcal{C})$$

defining $\mathcal{X}$ as an ∞-topos. By uniformization, and in particular keeping Example 4.7 in mind, we can find a cardinal κ such that

1. κ is 1-inaccessible;
2. $\kappa > |\mathcal{C}|$;
3. i and L both preserve κ-compact objects;
4. κ-compact objects in $\mathcal{X}$ are stable under finite limits.

By inaccessibility of κ, there is a Grothendieck universe $\mathcal{U}_\kappa \in \mathcal{U}$ associated to it. Condition (2) implies, by Proposition 2.4, that κ-compact objects in $\mathcal{P}(\mathcal{C})$ are exactly presheaves on $\mathcal{C}$ in the smaller universe. Condition (3) means that the adjunction above restricts to

$$\mathcal{X}^\kappa \underset{i_\kappa}{\overset{L_\kappa}{\leftrightarrows}} \mathcal{P}(\mathcal{C})^\kappa$$

and, moreover, L_κ preserves finite limits by condition (4). Therefore the restricted adjunction exhibits $\mathcal{X}^\kappa$ as a geometric ∞-topos in the smaller universe $\mathcal{U}_\kappa$. Again as in Proposition 7.3 and Proposition 7.4, this implies that $\mathcal{X}^\kappa$ is an elementary ∞-topos with strong universes in $\mathcal{U}$.

In this case, too, using Lemma 7.2 will ensure that $\mathcal{X}^\kappa$ cannot be a geometric ∞-topos in $\mathcal{U}$. ∎

A. Generic morphisms in geometric toposes

This appendix is dedicated to a proof of Proposition 4.17. A closely related statement is contained for example in [Str04]. The proof presented here follows the same general ideas, but it is more explicit and moreover, unless we are working in a presheaf topos, it proves a finer statement, in that we are pointing to precise classes with the required property.

A.1. Definition. *Given a small category $\mathcal{C}$, an object $C \in \mathcal{C}$ and a cardinal κ, a standard C-indexed κ-bounded presheaf on $\mathcal{C}$ is a presheaf $\mathcal{C}^{op} \to \mathbf{Set}$ of the form $D \mapsto \coprod_{f:D\to C} \lambda_f$ on objects, where λ_f is any cardinal $< \kappa$, regarded as a set, and such that the action on a morphism $g : D' \to D$ is given by the coproduct of maps of the form $\lambda_f \to \lambda_{fg}$.*

A.2. Remark. *The idea of the above definition is to specify very explicitly a collection of chosen representatives of equivalence classes of relatively κ-compact morphisms over $\mathbf{y}(C)$. Indeed, every standard C-indexed κ-bounded presheaf Q comes with a natural map $p_Q : Q \to \mathbf{y}(C)$ sending every element of λ_f to f.*

A.3. Lemma. *Given $C \in \mathcal{C}$ and κ as above, there is only a small set of standard C-indexed κ-bounded presheaves on $\mathcal{C}$.*

Proof. It suffices to count all the possible assignments on morphisms. Forgetting about the functoriality conditions and about the restriction on morphisms that is part of the definition, we will certainly obtain an upper bound for the quantity we are looking for. Thus, for a morphism $g : D' \to D$, the possible assignments are bounded by $|(\coprod_{h:D'\to C} \lambda_h)^{\coprod_{f:D\to C} \lambda_f}| \leq |(\mathcal{C}(D',C)| \cdot \kappa)^{|\mathcal{C}(D,C)| \cdot \kappa} \leq (|\mathcal{C}| \cdot \kappa)^{|\mathcal{C}| \cdot \kappa}$, which in fact in most cases is an overkill margin, and it is small because $\mathcal{C}$ is small by assumption. Now, since the number of morphisms in $\mathcal{C}$ is $|\mathcal{C}|$, all possible assignments are bounded by $|\mathcal{C}| \cdot (|\mathcal{C}| \cdot \kappa)^{|\mathcal{C}| \cdot \kappa}$. ∎

A.4. Lemma. *Given a standard C-indexed κ-bounded presheaf Q with its projection $p_Q : Q \to \mathbf{y}(C)$, then the set of sections of p_Q is in bijection with λ_{id_C}.*

Proof. We have the following natural equivalences

$$\mathrm{Sec}(p_Q) \cong \mathrm{Nat}(\mathbf{y}(C), Q) \times_{\mathrm{Nat}(\mathbf{y}(C),\mathbf{y}(C))} \{\mathrm{id}\} \cong Q(C) \times_{\mathcal{C}(C,C)} \{\mathrm{id}\} \cong \lambda_{id_C}$$

where the second isomorphism is given by Yoneda lemma, and the third by definition of standard C-indexed κ-bounded presheaf. ∎

A.5. Lemma. *Let $\mathcal{C}$ be a small category and let $\kappa > |\mathcal{C}|$ be a regular cardinal. Then a morphism $f : A \to B$ in $\mathcal{P}(\mathcal{C})$ is relatively κ-compact if and only if for every $C \in \mathcal{C}$ and every $x \in B(C)$ the fiber $B(C)_x$ is κ-small.*

Proof. Note that, by the analog of Proposition 2.4 for 1-categories, κ-compact presheaves are precisely those taking values in κ-small sets. Now choose a relatively κ-compact morphism $f : A \to B$ and express B as a κ-filtered colimit of κ-compact presheaves $B = \operatorname{colim} B_i$. Then by assumption on f any of the vertices F_i's of the pullback squares

$$\begin{array}{ccc} F_i & \longrightarrow & A \\ \big\downarrow & & \big\downarrow{\scriptstyle f} \\ B_i & \longrightarrow & B \end{array}$$

is κ-compact. By assumption on κ, this means that for every $C \in \mathcal{C}$, the set $F_i(C)$ is κ-small.

Now choose an object C and a point $x \in B(C)$. Since the singleton $*$ is κ-compact in **Set**, the map $x : * \to B(C)$ factors through one of the inclusions $B_i(C) \to B(C)$. By pasting law, we have that the outer square in the diagram

$$\begin{array}{ccccc} B_i(C)_x & \longrightarrow & F_i(C) & \longrightarrow & A(C) \\ \big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle f} \\ * & \xrightarrow{x} & B_i(C) & \longrightarrow & B(C) \end{array}$$

is a pullback, so that $B_i(C)_x \cong B(C)_x$, which is now seen to be a pullback of κ-small sets, which is itself κ-small by a set-theoretical argument.

Conversely, assume that every fiber $B(C)_x$ is κ-small. Let us choose a κ-compact presheaf Z and a morphism $g : Z \to B$. We want to show that $Z \times_B A$ is κ-compact, which amounts to proving that for each object C, the set $Z(C) \times_{B(C)} A(C)$ is κ-small. Now, for a point $y \in Z(C)$ there is a double pullback diagram

$$\begin{array}{ccccc} Z(C)_y & \longrightarrow & Z(C) \times_{B(C)} A(C) & \longrightarrow & A(C) \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ \{y\} & \longrightarrow & Z(C) & \xrightarrow{g} & B(C) \end{array}$$

which, by pasting law, implies that $Z(C)_y \cong B(C)_{g(y)}$, and this is κ-small by assumption. Finally, since $Z(C) \cong \coprod_{y \in Z(C)} \{y\}$, universality of colimits yields

$$Z(C) \times_{B(C)} A(C) \cong \coprod_{y \in Z(C)} Z(C)_y \cong \coprod_{y \in Z(C)} B(C)_{g(y)}$$

and this is a κ-small coproduct of κ-small sets, which is κ-small by regularity of κ. This concludes the proof. ∎

A.6. Proposition. *In a geometric topos, there are arbitrarily large regular cardinals κ such that the class S_κ of relatively κ-compact morphisms has a generic morphism.*

Proof. We will use the definition of geometric topos spelled out in Proposition 2.12(1). In **Set**, the statement is true for every cardinal κ, and a generic morphism is $\coprod_{\lambda<\kappa} \lambda \to \kappa$, where the cardinals are regarded as sets and the map sends every element of λ to λ regarded as an element of the set κ (compare, for example, with Example 6.4). Now, we will prove the statement in the case of presheaf toposes first, and finally in the general case.

Step 1. Given a small category $\mathcal{C}$, let us choose $\kappa > |\mathcal{C}|$, so that κ-compact presheaves $\mathcal{C}^{op} \to \mathbf{Set}$ are precisely those taking values in κ-small sets (analog of Proposition 2.4). Now let us define a presheaf U as follows: for every object $C \in \mathcal{C}$, $U(C)$ is the collection of standard C-indexed κ-bounded presheaves on $\mathcal{C}$, which is a set by Lemma A.3. For a morphism $h : C \to C'$, the map $U(C') \to U(C)$ is given as follows. If Q is of the form $Q(D) = \coprod_{f:D\to C'} \lambda_f$, then h^*Q will be of the form $h^*Q(D) = \coprod_{g:D\to C} \lambda'_g$, where $\lambda'_g = \lambda_{hg}$. It is easy to see that this is functorial.

Now, define another presheaf $\bar{U}$ as follows: for every object $C \in \mathcal{C}$, $\bar{U}(C)$ is the collection of pairs (Q, s) where Q is a standard C-indexed κ-bounded presheaf and s is a section of the projection p_Q. Lemma A.4 ensures that this is a set. Moreover, there is a natural map $t : \bar{U} \to U$, given by forgetting the sections. We want to show that this is a generic morphism for the class of relatively κ-compact morphisms. Note that in particular, when $\mathcal{C}$ is the terminal category, we recover the generic morphism given above.

Suppose that $A \to B$ is a relatively κ-compact natural transformation. By assumption on κ and Lemma A.5, this means that for every object C and every point $x \in B(C)$, the fiber $B(C)_x$ is κ-small. Choose bijections $\phi_x : B(C)_x \cong \lambda_x$ where $\lambda_x < \kappa$. Now define a natural transformation $\chi : B \to U$ as follows: if $x \in B(C)$, then $\chi(x)$ sends an object D to $\coprod_{f:D\to C} \lambda_{Bf(x)}$, and the action on morphisms is determined by that of A, using the chosen bijections ϕ_x. Let us fill the bullet in the double pullback diagram

$$\begin{array}{ccccc} F_x & \longrightarrow & \bullet & \longrightarrow & \bar{U}(C) \\ \downarrow & & \downarrow & & \downarrow t \\ \{x\} & \longrightarrow & B(C) & \xrightarrow{\chi} & U(C). \end{array}$$

An explicit computation yields that F_x is the set of all sections of $p_{\chi(x)}$ which, by Lemma A.4, is isomorphic to λ_{id_C}, but by definition of χ this is exactly λ_x. Moreover, we know that $\lambda_x \cong B(C)_x$. By universality of colimits in presheaf categories, we now have that

$$\bullet \cong \coprod_{x\in B(C)} F_x \cong \coprod_{x\in B(C)} \lambda_x \cong \coprod_{x\in B(C)} B(C)_x \cong A(C).$$

Notice that all these isomorphisms are natural in C, and in particular the third is so because the isomorphisms ϕ_x are functorial by construction. We then obtained that the functor $B \times_U \bar{U}$ is naturally isomorphic to A, so that $A \to B$ is seen to be a pullback of $t : \bar{U} \to U$.

Step 2. We will use Proposition 2.12(1). Given a geometric topos $\mathcal{X}$, let

$$\mathcal{X} \underset{i}{\overset{L}{\leftrightarrows}} \mathcal{P}(\mathcal{C})$$

be a left exact accessible localization, where $\mathcal{C}$ is a small category. We may assume that the Yoneda embedding $\mathcal{C} \hookrightarrow \mathcal{P}(\mathcal{C})$ factors through $\mathcal{X}$ (for example, by choosing $\mathcal{C} = \mathcal{X}^\lambda$, where λ is a cardinal such that $\mathcal{X}$ is λ-accessible. Using Example 4.7, let us choose a regular cardinal κ such that $\kappa > \max\{|\mathcal{C}|, \lambda\}$, all representables are κ-compact and i preserves κ-compact objects. We will first show that i preserves relatively κ-compact morphisms. Let $f : A \to B$ be relatively κ-compact in $\mathcal{X}$. Consider a κ-compact object Z in $\mathcal{P}(\mathcal{C})$ and a morphism $Z \to i(B)$. Express Z as a κ-small colimit of representables $(Z_i)_i$ and take the following double pullback diagram

$$\begin{array}{ccccc} Z'_i & \longrightarrow & Z' & \longrightarrow & i(A) \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle i(f)} \\ Z_i & \longrightarrow & Z & \longrightarrow & i(B). \end{array}$$

Now the outer square lives in $\mathcal{X}$ because all representables do and i preserves pullbacks. By assumption on f, Z'_i is κ-compact in $\mathcal{X}$, therefore also in $\mathcal{P}(\mathcal{C})$. By universality of colimits, we see that $Z' = \operatorname{colim}_i Z'_i$ and the indexing diagram is κ-small, therefore Z' is κ-compact by Proposition 2.1, which means that f is also relatively κ-compact when regarded as a morphism of $\mathcal{P}(\mathcal{C})$.

Now, by Step 1 we know that $\mathcal{P}(\mathcal{C})$ has a generic relatively κ-compact morphism. Let us call it $t : \bar{U} \to U$. We will show that Lt is a generic relatively κ-compact morphism in $\mathcal{X}$. Fix a relatively κ-compact morphism f. Since, as we have seen above, $i(f)$ is relatively κ-compact, there is a pullback square

$$\begin{array}{ccc} i(A) & \longrightarrow & \bar{U} \\ \downarrow{\scriptstyle i(f)} & & \downarrow{\scriptstyle t} \\ i(B) & \longrightarrow & U \end{array}$$

in $\mathcal{P}(\mathcal{C})$. Since L preserves pullbacks, this means that

$$\begin{array}{ccc} Li(A) & \longrightarrow & L(\bar{U}) \\ \downarrow{\scriptstyle Li(f)} & & \downarrow{\scriptstyle L(t)} \\ Li(B) & \longrightarrow & L(U) \end{array}$$

is a pullback square in $\mathcal{X}$. Since $Li(f) \cong f$, this concludes the proof. ∎

B. Small complete ∞-categories

This appendix is dedicated to a proof of Lemma 7.1, for which the strategy is to generalize the corresponding 1-categorical proof, already known and spelled out, for example, in [ML71], IV, Proposition 3.

In order to slightly simplify the proof of its ∞-categorical generalization and, moreover, to emulate more closely the proof of the classical statement, we will make use of a couple of definitions, whose intuitive idea is that we are looking at some selected cells of all possible dimensions in an ∞-category whose behaviour is reminiscent of cells in a strict higher category, where every n-cell has a domain and a codomain which are $(n-1)$-cells, so that it makes sense to say when two such cells are consecutive, what other cell (in our case, not uniquely determined) is a composite of them and to describe $(n+1)$-cells having two given n-cells as domain and codomain respectively.

We want to point out that it is not impossible to prove the theorem without the machinery hereafter presented, but we believe that, once said machinery has sunk in, it will be combinatorially simpler and, more than that, it should be clear that what we are doing is really just reproposing the classical proof in a higher dimensional form.

B.1. Definition. *Given a simplex σ of dimension n in a simplicial set, we know that there is a unique non-degenerate simplex ρ of dimension $m \leq n$ such that σ is a degeneracy of ρ. In this case, we will say that m is the essential dimension of σ.*

B.2. Definition. *Let $\mathcal{C}$ be an ∞-category. All simplices of dimension 0 or 1 are called 0-morphisms and 1-morphisms respectively. For $n \geq 2$, we say that $\alpha \in \mathcal{C}_n$ is an n-morphism if $\forall k = 0, \dots, n-2$ the k-th face of α is of essential dimension at most k. In this case, the $(n-1)$-th face of α will be called its codomain, and the n-th face its domain.*

B.3. Remark. *The condition on faces given in Definition B.2 should be interpreted as the simplex at issue being "as degenerate as possible" once given the last two faces, thus giving in practice no information other than a domain, a codomain and an arrow going from the former to the latter, just as n-morphisms in a strict higher category.*
Also, a brief inspection of an n-morphism in an ∞-category will reveal that both the domain and the codomain of an n-morphism are $(n-1)$-morphisms, and that they in turn have the same $(n-2)$-morphisms as domain and codomain. This allows us to talk about n-fold domains and codomains, obtained by taking either one n times, thus decreasing the dimension by n, just as it is done in strict higher categories or in globular sets.
The essential dimensions of the faces of an n-morphism should therefore be interpreted as carrying information about how degenerate the k-fold domains and codomains are.

The following construction is straightforward but very important for what follows. Given two objects X, Y in an ∞-category $\mathcal{C}$, the 0-simplices of $\mathrm{Hom}^L(X,Y)$ (defined for instance in [Lur09], Remark 1.2.2.5, but an equally valid discussion can be done using $\mathrm{Hom}^R(X,Y)$ instead) are precisely 1-morphisms having X as domain and Y as codomain, while 1-simplices from f to g are precisely 2-morphisms having f as domain and g as codomain. Now we can regard $\mathrm{Hom}^L(X,Y)$ as an ∞-category and look at $\mathrm{Hom}^L(f,g)$ whenever f

and g are 0-simplices of it. Similarly as above, 0-simplices of $\mathrm{Hom}^L(f,g)$ are 2-morphisms having f as domain and g as codomain, and given two such α and β, a 1-simplex from α to β is precisely a 3-morphism having α as domain and β as codomain.
Iterating this procedure, we obtain that for two n-morphisms ϕ and θ the n-fold Hom^L construction gives a mapping space $\mathrm{Hom}^L(\phi,\theta)$ whose 0-simplices are precisely $(n+1)$-morphisms from ϕ to θ and whose 1-simplices are precisely $(n+2)$-morphisms between those.

B.4. Definition. *Given two n-morphisms α and β in an ∞-category, we say that they are homotopic if there is a $(n+1)$-morphism having α as domain and β as codomain (in particular, α and β necessarily have the same domain and codomain themselves). As can be checked using horn fillings, this defines an equivalence relation on n-morphisms.*

B.5. Lemma. *Suppose $\mathcal{C}$ is an essentially small ∞-category, and that it has either all small limits or all small colimits. Then for any choice of objects $X,Y\in\mathcal{C}$ the mapping space* $\mathrm{Map}(X,Y)$ *is either empty or contractible.*

Proof. We will treat the case of completeness, the other is perfectly dual. Clearly the statement is invariant under equivalence, therefore we may assume that $\mathcal{C}$ is strictly small. We will prove that, as soon as $\mathrm{Map}(X,Y)$ is not empty, for $n\geq 0$ any map $\partial\Delta^n\to\mathrm{Map}(X,Y)$ can be extended to a map $\Delta^n\to\mathrm{Map}(X,Y)$. The case $n=0$ simply reaffirms that $\mathrm{Map}(X,Y)$ is not empty, therefore we may assume $n\geq 1$. Observe that if two Kan complexes are weakly equivalent, then one has the right lifting property against all inclusions $\partial\Delta^n\subset\Delta^n$ if and only the other does, because for Kan complexes this is the same as saying that the terminal map is a homotopy equivalence. Therefore, we are free to switch between different models for the mapping spaces involved, whenever it comes in handy. In particular, we will keep in mind the three homotopy equivalent objects mentioned in [Lur09], Corollary 4.2.1.8.
Assume by contradiction that there exists a map $q:\partial\Delta^n\to\mathrm{Hom}^L(X,Y)$ that admits no extension to Δ^n. This gives a non-null element of $\pi_{n-1}(\mathrm{Hom}^L(X,Y))$, which is isomorphic to $\pi_{n-2}(\Omega\mathrm{Hom}^L(X,Y))$, where the loop space is calculated taking $f=q(0)$ as base point. Therefore $\Omega\mathrm{Hom}^L(X,Y)=\mathrm{Fun}(\Delta^1,\mathrm{Hom}^L(X,Y))_{0,1\mapsto f}$. Using [Lur09], Corollary 4.2.1.8, we can replace this by $\mathrm{Hom}^L(f,f)$, therefore finding a non-null element of $\pi_{n-2}(\mathrm{Hom}^L(f,f))$. Repeating this procedure $n-1$ times, we obtain a non-null element of $\pi_0(\mathrm{Hom}^L(\phi,\phi))$, for some $(n-1)$-morphism ϕ. The fact that it is non-null means that there are two n-morphism that are not homotopic, in the sense of Definition B.4. In particular, we have proven that, once fixed a base point f, the elements of $\pi_n(\mathrm{Hom}^L(X,Y))$ are in bijection with homotopy classes of n-morphisms having a specific degeneracy of f as domain and codomain.
Now, let H_n be the quotient of the set of n-morphisms of $\mathcal{C}$ under the homotopy equivalence relation. Since this set is small and $\mathcal{C}$ is complete, we can consider the product $\prod_{\mathsf{H}_n}Y$, therefore we have an equivalence of mapping spaces $\mathrm{Hom}^L(X,\prod_{\mathsf{H}_n}Y)\simeq\prod_{\mathsf{H}_n}\mathrm{Hom}^L(X,Y)$. The discussion above says that there are at least two equivalence classes of n-morphisms having X and Y as n-fold domain and codomain, in particular

two distinct elements of $\pi_n(\mathrm{Hom}^L(X,Y))$, therefore, since the functor π_n preserves products, $\pi_n(\mathrm{Hom}^L(X,\prod_{\mathsf{H}_n} Y)) \cong \pi_n(\prod_{\mathsf{H}_n} \mathrm{Hom}^L(X,Y)) \cong \prod_{\mathsf{H}_n} \pi_n(\mathrm{Hom}^L(X,Y))$ has at least $2^{|\mathsf{H}_n|}$ distinct elements. Again the discussion above allows to reduce the analysis to equivalence classes of n-morphisms only, therefore giving an amount of such which is strictly bigger than the number of all equivalence classes of n-morphisms. We obtained thus the desired contradiction. ∎

References

[AR94] Jiří Adámek and Jiří Rosicky. *Locally Presentable and Accessible Categories.* Cambridge University Press, 1994.

[GHN15] David Gepner, Rune Haugseng, and Thomas Nikolaus. Lax colimits and free fibrations in ∞-categories. https://arxiv.org/abs/1501.02161v2, 2015.

[Jec06] Thomas Jech. *Set Theory.* Springer Verlag, 2006.

[LM18] Giulio Lo Monaco. On two extensions of the notion of ∞-topos. 2018.

[Lur09] Jacob Lurie. *Higher Topos Theory.* Princeton University Press, 2009.

[ML71] Saunders Mac Lane. *Categories for the Working Mathematician.* Springer Verlag, 1971.

[MP02] Ieke Moerdijk and Erik Palmgren. Type theories, toposes and constructive set theory: predicative aspects of AST. *Annals of Pure and Applied Logic*, (114):155–201, 2002.

[Pal91] Erik Palmgren. Transfinite hierarchies of universes, 1991.

[Pal98] Erik Palmgren. On universes in type theory. In *Twenty-five years of constructive type theory*, pages 191–204. Oxford University Press, 1998.

[Rat93] Michael Rathjen. The strength of some Martin-Löf type theories. https://www1.maths.leeds.ac.uk/~rathjen/typeOHIO.pdf, 1993.

[Rat00] Michael Rathjen. The strength of Martin-Löf type theory with a superuniverse. Part I. *Archive for Mathematical Logic*, (39):1–39, 2000.

[Rat01] Michael Rathjen. The strength of Martin-Löf type theory with a superuniverse. part II. *Archive for Mathematical Logic*, (40):207–233, 2001.

[RGP98] Michael Rathjen, Edward R. Griffor, and Erik Palmgren. Inaccessibility in constructive set theory and type theory. *Annals of Pure and Applied Logic*, (94):181–200, 1998.

[Shu12] Michael Shulman. Inductive and higher inductive types. https://home.sandiego.edu/~shulman/hottminicourse2012/04induction.pdf, 2012.

[Shu17] Michael Shulman. Elementary $(\infty, 1)$-topoi. https://golem.ph.utexas.edu/category/2017/04/elementary_1topoi.html, 2017.

[Shu19] Michael Shulman. All $(\infty,1)$-toposes have strict univalent universes. https://arxiv.org/abs/1904.07004, 2019.

[Str04] Thomas Streicher. Universes in toposes. http://www2.mathematik.tu-darmstadt.de/~streicher/NOTES/UniTop.pdf, 2004.

[VV.13] VV.AA. Homotopy Type Theory. Univalent Foundations of Mathematics. https://homotopytypetheory.org/book/, 2013.

Department of Mathematics and Statistics
Masaryk Univesrity, Brno, Czech Republic
Email: giulio.l.monaco@gmail.com

This article may be accessed at http://www.tac.mta.ca/tac/