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THE 2-NERVE OF A 2-GROUP AND DELIGNE'S DETERMINANT FUNCTORS

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Abstract. We prove that the bisimplicial set obtained by applying the 2-nerve functor of Lack and Paoli ([Lack, Paoli, 2008]) to a 2-group ([Baez, Lauda, 2004]) seen as a bicategory with one object, is a fibrant object in the universal simplicial replacement of Dugger ([Dugger, 2001]) of the model category of reduced homotopy 2-types. As an application we deduce a well known theorem about (non-symmetric) determinant functors for Waldhausen categories or derivators ([Deligne, 1987], [Knudsen, 2002], [Muro, Tonks, 2007] and [Muro, Tonks, Witte, 2015]).

1. Introduction

Let us denote by $\mathbf{K}^{\mathcal{A}}$ the reduced simplicial set associated to an exact category $\mathcal{A}$ by the $s_\bullet$-construction of [Waldhausen, 1983]. Thus an n-simplex of $\mathbf{K}^{\mathcal{A}}$ is a series of $n-1$ admissible monomorphisms of $\mathcal{A}$ together with choices of quotients. The homotopy groups of $\mathbf{K}^{\mathcal{A}}$ are the *Quillen K-theory groups* of $\mathcal{A}$, namely $K_n(\mathcal{A}) = \pi_{n+1}\big(\mathbf{K}^{\mathcal{A}}\big)$ for $n \geq 0$. Let is recall the following (non-stable) universal properties of the lower K-theory groups of $\mathcal{A}$:

- **K0** The group $K_0(\mathcal{A})$, namely the fundamental group of the reduced simplicial set $\mathbf{K}^{\mathcal{A}}$, represents the **Set**-valued functor of the *additive functions of* $\mathcal{A}$ defined in the category of (not necessarily commutative) groups $\mathbf{add}_{\mathcal{A}}\colon\ \mathbf{Grp} \longrightarrow \mathbf{Set}$; where an additive function of $\mathcal{A}$ with values in a group G is a function $\mathbf{a}\colon\ \mathrm{Ob}(\mathcal{A}) \longrightarrow G$ such that $\mathbf{a}(0) = e_G$ and verifying that $\mathbf{a}(B) = \mathbf{a}(A) \cdot \mathbf{a}(C)$ for every short exact sequence $A \rightarrowtail B \twoheadrightarrow C$ (see Section 4.1 below).

- **K1** The fundamental 2-group of the reduced simplicial set $\mathbf{K}^{\mathcal{A}}$, whose homotopy groups are the groups $K_0(\mathcal{A})$ and $K_1(\mathcal{A})$, represents the **Set**-valued functor of the (*homotopy classes of the*) *determinants of* $\mathcal{A}$ defined in the homotopy category of (not necessarily commutative) 2-groups $h\widetilde{\mathbf{det}}_{\mathcal{A}}(\,\cdot\,)\colon\ 2\text{-}h\mathbf{Grp} \longrightarrow \mathbf{Set}$ (see §4.6 of [Deligne, 1987] and Section 4.5 of this paper).

There is an enriched version of the property **K1**: Denote by $\mathbb{K}^{\mathcal{A}}$ the reduced bisimplicial set associated to $\mathcal{A}$ by the $w\mathcal{S}_\bullet$-construction of [Waldhausen, 1983]. Then a p-simplex of

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the simplicial set $\mathbb{K}^{\mathcal{A}}_{\bullet,q}$ is a p-chain of natural isomorphisms between elements of $\mathbf{K}^{\mathcal{A}}_q$. In particular the reduced simplicial sets $diag(\mathbb{K}^{\mathcal{A}})$ and $\mathbf{K}^{\mathcal{A}}$ are canonically weak homotopy equivalent. We have:

K1E The fundamental 2-group of the reduced simplicial set $diag(\mathbb{K}^{\mathcal{A}})$ represents the $h\mathbf{Grpd}$-valued functor $h\underline{\mathbf{det}}_{\mathcal{A}}(\,\cdot\,)\colon\ 2\text{-}h\mathbf{Grp} \longrightarrow h\mathbf{Grpd}$ of the (*functorial*) *determinants of* $\mathcal{A}$ defined in the homotopy category of (not necessarily commutative) 2-groups. Here $h\mathbf{Grpd}$ denotes the homotopy category of groupoids (see §4.3 of [Deligne, 1987] and Section 4.10 below).

In [Muro, Tonks, 2007] and [Muro, Tonks, Witte, 2015] there is a proof of the property **K1**E using quadratic modules and working for every reduced bisimplicial set, in particular for the bisimplicial set associated to a derivator as in [Maltsiniotis, 2007] or to a Waldhausen category by the $w\mathcal{S}_{\bullet}$-construction.

In this work, the universal property **K1**E that the fundamental 2-group of every reduced bisimplicial set has, will be deduced from the homotopy properties of the reduced bisimpicial set $\mathcal{N}^2(\mathcal{G})$ associated to a 2-group $\mathcal{G}$ by the 2-nerve functor of [Lack, Paoli, 2008], in the "canonical" simplicial model category structure on the category $\mathbf{ssSet}_0$ of reduced bisimplicial sets modeling the connected and pointed homotopy 2-type (see Proposition 2.9, Theorem 3.9 and Section 4.10).

The paper is structured as follows: In Section 2 we recall for every $0 \le n \le \infty$ two Quillen equivalent simplicial model categories modeling the connected and pointed homotopy n-types. To begin with we recall in Proposition 2.1 the simplicial model category structure on the category of reduced simplicial sets $\mathbf{sSet}_0$ where the cofibrations are the injections, the weak equivalences are the weak homotopy n-equivalences $\nu^{\star}\mathbf{W}_n$, and the mapping space is the restriction to $\mathbf{sSet}_0$ of the usual mapping space for pointed simplicial sets (see (2)). Then in Proposition 2.5 we give a generating set of trivial cofibrations for the family of fibrations between fibrants objects of the model category $(\mathbf{sSet}_0, \nu^{\star}\mathbf{W}_n)$.

On the other hand, applying the construction of [Dugger, 2001] to the model category $(\mathbf{sSet}_0, \nu^{\star}\mathbf{W}_n)$, we deduce in §2.8 a simplicial model category structure on the category of reduced bisimplicial sets $\mathbf{ssSet}_0$ where the cofibrations are the injections, the weak equivalences are the diagonal weak homotopy n-equivalences $d^{\star}\mathbf{W}_n$ and the mapping space is induced from the functor $p_1^{*}(\Delta^{\bullet})\colon\ \Delta \longrightarrow \mathbf{ssSet}_0$ defined by $p_1^{*}(\Delta^n)_{p,q} = \Delta^n_p$ (see (12)). In Corollary 2.10 we obtain sufficient conditions for a reduced bisimplicial set to be a fibrant object in the model category $(\mathbf{ssSet}_0, d^{\star}\mathbf{W}_n)$.

Section 3 begins by fixing some notations about the 2-category $\underline{\mathbf{cat}}^{\otimes}_{Nlax}$ of monoidal categories, normal lax monoidal functors and monoidal natural transformations, as well as the full 2-subcategory $2\text{-}\underline{\mathbf{Grp}}$ of $\underline{\mathbf{cat}}^{\otimes}_{Nlax}$ whose objects are the 2-groups. Then in §3.5 we define the geometrical nerve functor $\mathcal{N}\colon\ \mathbf{cat}^{\otimes}_{Nlax} \longrightarrow \mathbf{sSet}_0$ as in [Street, 1987], and we prove in Lemma 3.6 that $\mathcal{N}$ is the same as the nerve functor of [Duskin, 2002] when restricted to bicategories with one object. In Corollary 3.7 we recall that the geometrical nerve of a 2-group $\mathcal{N}(\mathcal{G})$ is a fibrant object of the model category $(\mathbf{sSet}_0, \nu^{\star}\mathbf{W}_2)$.

In §3.8 we define a geometrical nerve $\mathbf{ssSet}_0$-valued functor $\mathcal{N}^2\colon \mathbf{cat}^{\otimes}_{Nlax} \longrightarrow \mathbf{ssSet}_0$ taking a "simplicial resolution" of the functor $\mathcal{N}$. Then we prove in Theorem 3.9 that $\mathcal{N}^2(\mathcal{G})$ is a fibrant object of the model category $(\mathbf{ssSet}_0, d^\star\mathbf{W}_2)$ for every 2-group $\mathcal{G}$. Moreover we proof in Lemma 3.11 that the functor $\mathcal{N}^2$ is full and faithful and in §3.12 that $\mathcal{N}^2$ has an extension to a (full and faithful) simplicial functor (see Corollary 3.14).

We finish the paper with Section 4 where we deduce the properties **K1** and **K1**E (resp. the property **K0**) from the homotopy properties of the geometrical nerves of a 2-group $\mathcal{N}(\mathcal{G})$ and $\mathcal{N}^2(\mathcal{G})$ (resp. the geometrical nerve of a group $\mathrm{N}(G)$) in the corresponding simplicial model category $(\mathbf{sSet}_0, \nu^\star\mathbf{W}_2)$ or $(\mathbf{ssSet}_0, d^\star\mathbf{W}_2)$ $\big($resp. $(\mathbf{sSet}_0, \nu^\star\mathbf{W}_1)\big)$.

2. Reduced homotopy n-types

Let **sSet** be the category of simplicial sets (see [May, 1967] or [Goerss, Jardine, 1999]). A *reduced simplicial set* is a simplicial set X with exactly one 0-simplex. Denote by $\mathbf{sSet}_0$ the full subcategory of **sSet** whose objects are the reduced simplicial sets. If $\mathbf{sSet}_\star$ denote the category of pointed simplicial sets, then the inclusion functor $\nu\colon \mathbf{sSet}_0 \longrightarrow \mathbf{sSet}$ canonically decomposes as the composition of a functor $\mu\colon \mathbf{sSet}_0 \longrightarrow \mathbf{sSet}_\star$ and the forgetful functor $\pi\colon \mathbf{sSet}_\star \longrightarrow \mathbf{sSet}$. Notice that there are adjunctions:

$$\begin{tikzcd}
\mathbf{sSet}_0 \arrow[r, hook, "\nu"'] \arrow[r, phantom, shift left=2, "\perp"] & \mathbf{sSet} \arrow[l, bend right=40, "\mathcal{F}"'] & \text{and} & \mathbf{sSet}_0 \arrow[r, hook, "\mu" description] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] & \mathbf{sSet}_\star\,, \arrow[l, bend right=40, "\mathcal{G}"'] \arrow[l, bend left=40, "\mathcal{H}"]
\end{tikzcd} \tag{1}$$

where $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ are defined as follows: If X is a simplicial set and $x_0 \in X_0$, then the reduced simplicial set $\mathcal{G}(X, x_0) = \mathcal{F}(X)$ is the quotient $X\big/X_0$ whose n-simplices are obtained from X_n by identifying all the totally degenerated n-simplices in one point; and $\mathcal{H}(X, x_0)_n$ is the set of those n-simplices of X whose 0-vertex are all equal to x_0.

We deduce that $\mathbf{sSet}_0$ is a complete and a cocomplete category. In fact given a small category I and a functor $\gamma\colon I \longrightarrow \mathbf{sSet}_0$ we have: (i) A limit of γ in the category **sSet** is a limit of γ in the category $\mathbf{sSet}_0$. (ii) If $\operatorname*{colim}_I \gamma$ is a colimit of γ in **sSet**, a quotient $\big(\operatorname*{colim}_I \gamma\big)\Big/\big(\operatorname*{colim}_I \gamma\big)_0$ in the category **sSet** is a colimit of γ in $\mathbf{sSet}_0$. In particular if I is connected, a colimit of γ in **sSet** is a colimit of γ in $\mathbf{sSet}_0$.

For a reduced simplicial set X and a natural number $i \geq 1$ we denote by $\pi_i(X)$ the i-th homotopy group of the geometric realization of the canonically pointed simplicial set $\mu(X) = X$. As usual this groups can be combinatorially defined for *reduced Kan complexes*, *i.e.* reduced simplicial sets satisfying the Kan condition. Consider $0 \leq n \leq \infty$, a map of reduced simplicial sets $f\colon X \longrightarrow Y$ is called a *weak homotopy n-equivalence* if the induced map $f_*\colon \pi_i(X) \longrightarrow \pi_i(Y)$ is a group isomorphism for $1 \leq i \leq n$.

We shall outline a proof of the next statement:

2.1. Proposition. *For $0 \leq n \leq \infty$, the category* $\mathbf{sSet}_0$ *of reduced simplicial sets support a left proper and combinatorial model category structure denoted by* $\big(\mathbf{sSet}_0, \nu^\star \mathbf{W}_n\big)$ *where the weak equivalences are the weak homotopy n-equivalences and the cofibrations are the injections. A reduced simplicial set X is a fibrant object of* $\big(\mathbf{sSet}_0, \nu^\star \mathbf{W}_n\big)$ *if and only if X is a Kan complex such that $\pi_i\big(X\big) = 0$ for every $i \geq n+1$.*

Moreover $\big(\mathbf{sSet}_0, \nu^\star \mathbf{W}_n\big)$ *is a simplicial model category with the enrichment:*

$$\underline{\mathrm{Hom}}_{\mathbf{sSet}_0}(X,Y)_k \;=\; \mathrm{Hom}_{\mathbf{sSet}_0}(X \wedge \Delta^k_+, Y) \;=\; \mathrm{Hom}_{\mathbf{sSet}_\star}(X \wedge \Delta^k_+, Y) \;=\; \underline{\mathrm{Hom}}_{\mathbf{sSet}_\star}(X,Y)_k\,, \tag{2}$$

where $X \wedge K_+ = (X \times K)\big/(\star \times K)$ is a reduced simplicial set for every simplicial set K.

In Proposition 6.2 of [Goerss, Jardine, 1999] there is a proof that $\big(\mathbf{sSet}_0, \nu^\star \mathbf{W}_\infty\big)$ is a model category. It is easy to see (Lemma 6.6 of *loc.cit.*) that a map of reduced simplicial sets $f\colon\ X \longrightarrow Y$ is a Kan fibration, if and only if f is a fibration in the model category $\big(\mathbf{sSet}_0, \nu^\star \mathbf{W}_\infty\big)$ and f satisfies the right lifting property with respect to the map $\star \longrightarrow \mathbb{S}^1 = \Delta^1/\partial\Delta^1$. In particular a reduced simplicial set X is a fibrant object of $\big(\mathbf{sSet}_0, \nu^\star \mathbf{W}_\infty\big)$ if and only if X is a Kan complex.

Notice that the adjunction $\mathcal{F} \dashv \nu$ of (1) proof that $\mathbf{sSet}_0$ is equivalent to a full reflective subcategory of a presheaf category $\mathbf{Set}^{\mathcal{A}}$ which is closed under small direct colimits (or equivalently under small filtrants colimits), so $\mathbf{sSet}_0$ is a locally presentable category (Proposition 1.46 of [Adamek, Rosicky, 1994]).

On the other hand the classical proof that every map $\varphi\colon\ X \longrightarrow Y$ of $\mathbf{sSet}$ is an injection if and only if φ is the transfinite composition of direct images of maps of the form $\partial\Delta^n \hookrightarrow \Delta^n$ where $n \geq 0$ (see for example §2.1 of [Hovey, 1999]), can be slightly modified to show that a map $\varphi\colon\ X \longrightarrow Y$ of reduced simplicial sets is an injection if and only if φ is the transfinite composition of direct images (in the category $\mathbf{sSet}_0$) of maps of the form $\partial\Delta^n/\partial\Delta^n_0 \hookrightarrow \Delta^n/\Delta^n_0$ where $n \geq 1$. Then $\big(\mathbf{sSet}_0, \nu^\star \mathbf{W}_\infty\big)$ is a cofibrantly generated model category.

Moreover from the canonical simplicial model category structure on the category of pointed simplicial sets in which the weak eqivalences are the weak homotopy equivalences between pointed simplicial sets and the cofibrations are the injections (see §4.2 of [Hovey, 1999]), we could deduce that $\big(\mathbf{sSet}_0, \nu^\star \mathbf{W}_\infty\big)$ is a simplicial model category with the enrichment (2).

Indeed note that we have the adjunctions:

$$\begin{tikzcd} \mathbf{sSet}_0 \arrow[r, bend right, "\underline{\mathrm{Hom}}_{\mathbf{sSet}_0}(X,\,\cdot\,)"'] \arrow[r, phantom, "\perp"] & \mathbf{sSet} \arrow[l, bend right, "X\wedge(\,\cdot\,)_+"'] & \text{and} & \mathbf{sSet}_0^{op} \arrow[r, bend right, "\underline{\mathrm{Hom}}_{\mathbf{sSet}_0}(\,\cdot\,,Y)"'] \arrow[r, phantom, "\perp"] & \mathbf{sSet} \arrow[l, bend right, "Y^{\wedge\,\cdot}"'] \end{tikzcd} \tag{3}$$

where $\big(Y^{\wedge K}\big)_k \;=\; \mathcal{H}\big(\underline{\mathrm{Hom}}_{\mathbf{sSet}_*}(K_+, Y), \star\big)_k \;=\; \mathrm{Hom}_{\mathbf{sSet}_0}\big((K \times \Delta^k)\big/(K \times \Delta^k_0), Y\big)$ for $k \geq 0$.

Furthermore given an injection of simplicial sets $A \xrightarrow{j} B$ and an injection of reduced simplicial sets $X \xrightarrow{q} Y$ consider the following pushout diagram in the category $\mathbf{ssSet}_0$:

$$\begin{tikzcd}
X \wedge A_+ \arrow[r, "q\wedge A_+"] \arrow[d, "X\wedge j_+"'] & Y \wedge A_+ \arrow[d] \arrow[ddr, bend left, "Y\wedge j_+"] & \\
X \wedge B_+ \arrow[r] \arrow[drr, bend right, "q\wedge B_+"'] & X \wedge B_+ \coprod\limits_{X\wedge A_+} Y \wedge A_+ \arrow[dr, dashed, "\varphi"] & \\
 & & Y \wedge B_+ .
\end{tikzcd} \tag{4}$$

Because the functor μ of (1) commute with small limits and colimits we deduce that the dashed arrow φ of the diagram (4) is an injection, which is also a weak homotopy ∞-equivalence if j or q are weak homotopy ∞-equivalences.

Hence $\big(\mathbf{sSet}_0, \nu^\star \mathbf{W}_\infty\big)$ is a left proper, combinatorial and simplicial model category. For $0 \leq n < \infty$ we want to exhibit the model category $\big(\mathbf{sSet}_0, \nu^\star \mathbf{W}_n\big)$ as the Bousfield localization of $\big(\mathbf{sSet}_0, \nu^\star \mathbf{W}_\infty\big)$ with respect to the set of maps of reduced simplicial sets $S(n) = \big\{ \, \mathbb{S}^i \colon = \Delta^i/\partial\Delta^i \longrightarrow \Delta^0 \, \big\}_{i \geq n+1}$.

2.2. Lemma. *A reduced simplicial set Z is an $S(n)$-local object of $\big(\mathbf{sSet}_0, \nu^\star \mathbf{W}_\infty\big)$ if and only if $\pi_i(Z) = 0$ for all $i \geq n+1$.*

Proof. Let Z be a reduced simplicial set and $Z' \longrightarrow Z$ a fibrant replacement of Z in $\big(\mathbf{sSet}_0, \nu^\star \mathbf{W}_\infty\big)$. Notice that in this case Z' and $\underline{\mathrm{Hom}}_{\mathbf{sSet}_0}(\mathbb{S}^i, Z') = \underline{\mathrm{Hom}}_{\mathbf{sSet}_\star}(\mathbb{S}^i, Z')$ are Kan complexes in particular we have the isomorphisms:

$$\begin{aligned}
\pi_k \left(\underline{\mathrm{Hom}}_{\mathbf{sSet}_\star}(\mathbb{S}^i, Z') \right) &\cong \pi_0 \left(\underline{\mathrm{Hom}}_{\mathbf{sSet}_\star}\left(\mathbb{S}^k, \underline{\mathrm{Hom}}_{\mathbf{sSet}_\star}(\mathbb{S}^i, Z')\right)\right) \\
&\cong \pi_0 \left(\underline{\mathrm{Hom}}_{\mathbf{sSet}_\star}(\mathbb{S}^{k+i}, Z') \right) \cong \pi_{k+i}(Z') \cong \pi_{k+i}(Z) \, .
\end{aligned}$$

where the simplicial set $\underline{\mathrm{Hom}}_{\mathbf{sSet}_\star}(\mathbb{S}^i, Z')$ is pointed by the constant map.

By definition Z is an $S(n)$-local object of $\big(\mathbf{sSet}_0, \nu^\star \mathbf{W}_\infty\big)$ if and only if the induced map of simplicial sets:

$$\underline{\mathrm{Hom}}_{\mathbf{sSet}_\star}(\mathbb{S}^i, Z') \; = \; \underline{\mathrm{Hom}}_{\mathbf{sSet}_0}(\mathbb{S}^i, Z') \longrightarrow \underline{\mathrm{Hom}}_{\mathbf{sSet}_0}(\Delta^0, Z') \cong \Delta^0$$

is a weak homotopy equivalence for all $i \geq n+1$. Equivalently if for all $i \geq n+1$ and $k \geq 0$ we have that $\pi_{k+i}(Z) \; \cong \; \pi_k \left(\underline{\mathrm{Hom}}_{\mathbf{sSet}_\star}(\mathbb{S}^i, Z') \right) \; = \; 0$. ■

If $n \geq -1$ is a natural number denote by $\Delta_{\leq n+1}$ the full subcategory of the simplex category Δ whose objects are the totally ordered sets $[k] = \{0 < \cdots < k\}$ for $0 \leq k \leq n+1$. The inclusion functor $\tau_{n+1} \colon \; \Delta_{\leq n+1} \hookrightarrow \Delta$ induce an adjunction:

$$\begin{tikzcd}
\mathbf{sSet}_{\leq n+1} \arrow[r, bend right, "{\tau_{n+1}}_*"'] \arrow[r, phantom, "\perp"] & \mathbf{sSet} \arrow[l, bend right, "\tau^*_{n+1}"']
\end{tikzcd} \tag{5}$$

where $\mathbf{sSet}_{\leq n+1}$ is the category of presheaves of sets over $\Delta_{\leq n+1}$. The functor τ^*_{n+1} is just the truncation functor and the composition $\tau_{n+1*} \circ \tau^*_{n+1} = \mathrm{csk}_{n+1}$ is the called $(n+1)$-coskeleton functor.

Let $\eta\colon \mathrm{id}_{\mathbf{sSet}} \Rightarrow \mathrm{csk}_{n+1}$ be a fixed unit of the adjunction (5). It can be shown that if Z is a reduced Kan complex then $\mathrm{csk}_{n+1}(Z)$ is also a reduced Kan complex such that $\pi_i\big(\mathrm{csk}_{n+1}(Z)\big) = 0$ for $i \geq n+1$ and the map $\eta_Z\colon\ Z \longrightarrow \mathrm{csk}_{n+1}(Z)$ is a weak homotopy n-equivalence (see [May, 1967], [Duskin, 2002] or [Cisinski, 2006]). In particular a map f of reduced Kan complexes is a weak homotopy n-equivalence if and only if $\mathrm{csk}_{n+1}(f)$ is a weak homotopy ∞-equivalence.

It follows from Lemma 2.2 that if Z is a reduced Kan complex, then Z is an $S(n)$-local object of $\big(\mathbf{sSet}_0, \nu^\star\mathbf{W}_\infty\big)$ if and only if the map $\eta_Z\colon\ Z \longrightarrow \mathrm{csk}_{n+1}(Z)$ is a weak homotopy ∞-equivalence.

Notice finally that for Z and W arbitrary reduced simplicial sets the map:

$$\underline{\mathrm{Hom}}_{\mathbf{sSet}_0}\big(W, \mathrm{csk}_{n+1}(Z)\big) \longrightarrow \underline{\mathrm{Hom}}_{\mathbf{sSet}_0}\big(\mathrm{csk}_{n+1}(W), \mathrm{csk}_{n+1}(Z)\big) \tag{6}$$

induced from $\eta_W\colon\ W \longrightarrow \mathrm{csk}_{n+1}(W)$ is an isomorphism of simplicial sets.

Indeed the truncation functor τ^*_{n+1} commutes with small colimis and limits, so for $k \geq 0$ we have the natural isomorphisms:

$$\begin{aligned}\underline{\mathrm{Hom}}_{\mathbf{sSet}_0}\Big(W,\mathrm{csk}_{n+1}(Z)\Big)_k &= \mathrm{Hom}_{\mathbf{sSet}_0}\Big(W\wedge\Delta^k_+,\mathrm{csk}_{n+1}(Z)\Big) = \mathrm{Hom}_{\mathbf{sSet}}\Big(W\wedge\Delta^k_+,\mathrm{csk}_{n+1}(Z)\Big)\\ &\cong \mathrm{Hom}_{\mathbf{sSet}_{\leq n+1}}\Big(\tau^*_{n+1}(W)\wedge\tau^*_{n+1}(\Delta^k)_+,\tau^*_{n+1}(Z)\Big)\end{aligned}$$

and

$$\begin{aligned}\underline{\mathrm{Hom}}_{\mathbf{sSet}_0}\Big(\mathrm{csk}_{n+1}(W),\mathrm{csk}_{n+1}(Z)\Big)_k &= \mathrm{Hom}_{\mathbf{sSet}_0}\Big(\mathrm{csk}_{n+1}(W)\wedge\Delta^k_+,\mathrm{csk}_{n+1}(Z)\Big) = \mathrm{Hom}_{\mathbf{sSet}}\Big(\mathrm{csk}_{n+1}(W)\wedge\Delta^k_+,\mathrm{csk}_{n+1}(Z)\Big)\\ &\cong \mathrm{Hom}_{\mathbf{sSet}_{\leq n+1}}\Big(\tau^*_{n+1}(\mathrm{csk}_{n+1}(W))\wedge\tau^*_{n+1}(\Delta^k)_+,\tau^*_{n+1}(Z)\Big)\ .\end{aligned}$$

In the same way it is possible to proof that the map:

$$\underline{\mathrm{Hom}}_{\mathbf{sSet}_0}\big(W, \mathrm{csk}_{n+1}(Z)\big) \xrightarrow{\eta_{\underline{\mathrm{Hom}}(W,\mathrm{csk}Z)}} \mathrm{csk}_{n+1}\Big(\underline{\mathrm{Hom}}_{\mathbf{sSet}_0}\big(W, \mathrm{csk}_{n+1}(Z)\big)\Big) \tag{7}$$

is also an isomorphism of simplicial sets.

2.3. Lemma. *A map $f\colon\ X \longrightarrow Y$ of reduced simplicial sets is an $S(n)$-local equivalence of the model category $\big(\mathbf{sSet}_0, \nu^\star\mathbf{W}_\infty\big)$ if and only if f is a weak homotopy n-equivalence.*

Proof. Let $f\colon\ X \longrightarrow Y$ be a map of reduced simplicial sets. We can suppose that X and Y are Kan complexes. According to the isomorphism (6) it follows that f is an $S(n)$-local equivalence of the model category $\big(\mathbf{sSet}_0, \nu^\star\mathbf{W}_\infty\big)$ if and only if, for every reduced Kan complex Z the map of simplicial sets:

$$\underline{\mathrm{Hom}}_{\mathbf{sSet}_0}\big(\mathrm{csk}_{n+1}(Y), \mathrm{csk}_{n+1}(Z)\big) \xrightarrow{\mathrm{csk}_{n+1}(f)^*} \underline{\mathrm{Hom}}_{\mathbf{sSet}_0}\big(\mathrm{csk}_{n+1}(X), \mathrm{csk}_{n+1}(Z)\big) \tag{8}$$

is a weak homotopy ∞-equivalence.

If the map f is a weak homotopy n-equivalence we have that $\mathrm{csk}_{n+1}(f)$ is a weak homotopy ∞-equivalence between reduced Kan complexes, then (8) is a weak homotopy ∞-equivalence for every reduced Kan complex Z *i.e.* f is an $S(n)$-local equivalence.

Reciprocally suppose f is an $S(n)$-local equivalenced of $\big(\mathbf{sSet}_0, \nu^\star\mathbf{W}_\infty\big)$. We deduce from the weak homotopy ∞-equivalence (8) for $Z = X$ that there exist a map $g\colon\ \mathrm{csk}_{n+1}(Y) \longrightarrow \mathrm{csk}_{n+1}(X)$ of reduced simplicial sets such that the composition $g \circ \mathrm{csk}_{n+1}(f)$ is equal to the identity map of the object $\mathrm{csk}_{n+1}(X)$ in the homotopy category $\mathbf{sSet}_0\big[(\nu^\star\mathbf{W}_\infty)^{-1}\big]$. In particular the functions:

$$\big[\mathrm{csk}_{n+1}(Y), \mathrm{csk}_{n+1}(X)\big]^{red}_\infty \underset{g^*}{\overset{\mathrm{csk}_{n+1}(f)^*}{\rightleftarrows}} \big[\mathrm{csk}_{n+1}(X), \mathrm{csk}_{n+1}(X)\big]^{red}_\infty\,,$$

where $\big[\ ,\ \big]^{red}_\infty$ denote the set of morphisms in the homotopy category $\mathbf{sSet}_0\big[(\nu^\star\mathbf{W}_\infty)^{-1}\big]$, verify that the function $\mathrm{csk}_{n+1}(f)^*$ is a bijection and the function g^* is the right inverse of $\mathrm{csk}_{n+1}(f)^*$. Then g^* is a also the left inverse of $\mathrm{csk}_{n+1}(f)^*$ so the composition $\mathrm{csk}_{n+1}(f)\circ g$ is equal to the identity map of the object $\mathrm{csk}_{n+1}(Y)$ in the homotopy category $\mathbf{sSet}_0\big[(\nu^\star\mathbf{W}_\infty)^{-1}\big]$.

Therefore $\mathrm{csk}_{n+1}(f)$ is a weak homotopy ∞-equivalence *i.e.* f is a weak homotopy n-equivalence. ■

Notice that for $0 \le n \le \infty$ the adjunction $\mu \dashv \mathcal{H}$ in (1) is a Quillen adjunction between the model category $\big(\mathbf{sSet}_0, \nu^\star\mathbf{W}_n\big)$ of the Proposition 2.1 and the model category $(\mathbf{sSet}_\star, \pi^\star\mathbf{W}_n)$ whose underlying category is the category of pointed simplicial sets, the weak equivalences are the weak homotopy n-equivalences and the cofibrations are the injections (see [Cisinski, 2006] or [Biedermann, 2008] for the unpointed model category).

Furthermore it is not difficult to see that the induced adjunction:

$$\begin{tikzcd} \mathbf{sSet}_0\big[(\nu^\star\mathbf{W}_n)^{-1}\big] \arrow[r, bend left, "\mathbf{L}\mu"] \arrow[r, phantom, "\perp"] & \mathbf{sSet}_\star\big[(\pi^\star\mathbf{W}_n)^{-1}\big] \arrow[l, bend left, "\mathbf{R}\mathcal{H}"] \end{tikzcd} \tag{9}$$

induce in turn an equivalence between the category $\mathbf{sSet}_0\big[(\nu^\star\mathbf{W}_n)^{-1}\big]$ and the full subcategory of the category $\mathbf{sSet}_\star\big[(\nu^\star\mathbf{W}_n)^{-1}\big]$ whose objects are the connected and pointed simplicial sets.

2.4. In this paragraph we shall prove the next characterization of the fibrations between fibrant objects of the model category of the Proposition 2.1:

2.5. Proposition. *Let* $0 \le n \le \infty$. *In the model category* $\big(\mathbf{sSet}_0, \nu^\star\mathbf{W}_n\big)$ *a map* $f\colon\ X \longrightarrow Y$ *between fibrant objects is a fibration, if and only if it satisfies the right lifting property with respect to the maps of the set:*

$$J_0 = \left\{\quad \Lambda^{m,k}/\Lambda^{m,k}_0 \hookrightarrow \Delta^m/\Delta^m_0 \quad \middle|\quad m \ge 2\,,\ \ 0 \le k \le m \quad\right\}. \tag{10}$$

Since $\left(\mathbf{sSet}_0, \nu^\star\mathbf{W}_n\right)$ is a left Bousfield localisation of $\left(\mathbf{sSet}_0, \nu^\star\mathbf{W}_\infty\right)$, it follows from Proposition 3.4.16 of [Hirschhorn, 2002] that a map $f\colon\ X \to Y$ between two fibrant objects of $\left(\mathbf{sSet}_0, \nu^\star\mathbf{W}_n\right)$ is a fibration of $\left(\mathbf{sSet}_0, \nu^\star\mathbf{W}_n\right)$ if and only if it is a fibration of $\left(\mathbf{sSet}_0, \nu^\star\mathbf{W}_\infty\right)$.

So to prove the Proposition 2.5 it suffices to show that a map $f\colon\ X \longrightarrow Y$ between reduced Kan complexes is a fibration of $\left(\mathbf{sSet}_0, \nu^\star\mathbf{W}_\infty\right)$ if and only if it verifies the right lifting property with respect to the maps in the set (10).

Let $\mathcal{A}$ be the set of maps of reduced simplicial sets $f\colon\ X \longrightarrow Y$ such that X and Y are Kan complexes and f verifies the right lifting property with respect to the maps in the set J_0 of (10). It follows by a standard method (see the Lemma 2.1 of [Stanculescu, 2014]) that to get our aim we just need to show that the elements of J_0 are trivial cofibrations of $\left(\mathbf{sSet}_0, \nu^\star\mathbf{W}_\infty\right)$ and that the maps in $\mathcal{A} \cap \nu^\star\mathbf{W}_\infty$ are fibrations of $\left(\mathbf{sSet}_0, \nu^\star\mathbf{W}_\infty\right)$.

2.6. Lemma. *The maps in* (10) *are injections and homotopy weak* ∞*-equivalences.*

Proof. Clearly the induced map $A/A_0 \longrightarrow B/B_0$ is an injection of reduced simplicial sets for every injection of simplicial sets $A \longrightarrow B$. On the other hand consider for $m \geq 2$ and $0 \leq k \leq m$ the cube:

$$\begin{tikzcd}
& \Lambda_0^{m,k} \arrow[rr] \arrow[dl] \arrow[dd, equal] & & \Lambda^{m,k} \arrow[dl] \arrow[dd] \\
\star \arrow[rr, crossing over] \arrow[dd, equal] & & \Lambda^{m,k}/\Lambda_0^{m,k} & \\
& \Delta_0^m \arrow[rr] \arrow[dl] & & \Delta^m \arrow[dl] \\
\star \arrow[rr] & & \Delta^m/\Delta_0^m \arrow[from=uu, crossing over] &
\end{tikzcd}$$

and notice that the top and bottom pushout squares are actually homotopy pushout squares in the usual model category of simplicial sets $(\mathbf{sSet}, \mathbf{W}_\infty)$, where the weak equivalences are the weak homotopy ∞-equivalences and the cofibrations are the injections. Indeed the canonical map $A_0 \longrightarrow A$ is always a injection of simplicial sets (see for example the Proposition A.2.4.4 of [Lurie, 2009]).

We conclude that for $m \geq 2$ and $0 \leq k \leq m$ the map $\Lambda^{m,k}/\Lambda_0^{m,k} \hookrightarrow \Delta^m/\Delta_0^m$ is a weak homotopy ∞-equivalence because $\Lambda^{m,k} \hookrightarrow \Delta^m$ is a weak homotopy ∞-equivalence for $m \geq 1$ and $0 \leq k \leq m$. ■

Notice that the trivial fibrations of $\left(\mathbf{sSet}_0, \nu^\star\mathbf{W}_\infty\right)$ between Kan complexes are actually Kan fibrations (see the Corollary 6.9 of [Goerss, Jardine, 1999]). We shall then prove:

2.7. Lemma. *Let* $f\colon\ X \longrightarrow Y$ *be a map in* $\mathcal{A}$*. If the induced function of fundamental groups* $f_\star\colon\ \pi_1(X) \longrightarrow \pi_1(Y)$ *is surjective, then* f *is a Kan fibration.*

Proof. We just need to prove that f verifies the right lifting property with respect to the map $\star \hookrightarrow \Delta^1/\Delta^1_0$; that is to say that the function $f_1\colon\ X_1 \longrightarrow Y_1$ is surjective.

Let $y \in Y_1$. We deduce from the combinatorial definition of the fundamental group of a Kan complex and the surjectivity of the function $f_\star\colon\ \pi_1(X) \longrightarrow \pi_1(Y)$ that there exist $x \in X_1$ and $\eta \in Y_2$ such that $d_2(\eta) = f_1(x)$, $d_1(\eta) = y$ and $d_0(\eta) = \star$.

Consider the only map of simplicial sets $\eta'\colon\ \Lambda^{2,1}/\Lambda^{2,1}_0 \longrightarrow X$ verifying the conditions:

$$\begin{tikzcd}
\Delta^1 \arrow[r, "\delta_2"'] \arrow[rrr, bend left, "x"] & \Lambda^{2,1} \arrow[r, "\text{quotient}"'] & \Lambda^{2,1}/\Lambda^{2,1}_0 \arrow[r, "\eta'"'] & X
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\Delta^1 \arrow[r, "\delta_0"'] \arrow[rrr, bend left, "\star"] & \Lambda^{2,1} \arrow[r, "\text{quotient}"'] & \Lambda^{2,1}/\Lambda^{2,1}_0 \arrow[r, "\eta'"'] & X
\end{tikzcd} ;$$

so we have a commutative square:

$$\begin{tikzcd}
\Lambda^{2,1}/\Lambda^{2,1}_0 \arrow[r, "\eta'"] \arrow[d] & X \arrow[d, "f"] \\
\Delta^2/\Delta^2_0 \arrow[r, "\eta"'] & Y .
\end{tikzcd} \tag{11}$$

Let $\xi\colon\ \Delta^2/\Delta^2_0 \longrightarrow X$ be a lifting of the square (11). We deduce that $f_1(d_1\xi) = d_1(f_2\xi) = d_1(\eta) = y$. ■

2.8. Let **ssSet** be the category of bisimplicial sets (see for exemple [Goerss, Jardine, 1999]). A *reduced bisimplicial set* is a bisimplicial set X such that $X_{p,\bullet}$ is a reduced simplicial set for every $p \geq 0$. Denote by $\mathbf{ssSet}_0$ the full subcategory of **ssSet** whose objects are the reduced bisimplicial sets.

If $0 \leq n \leq \infty$ we call a map $f\colon\ X \longrightarrow Y$ in $\mathbf{ssSet}_0$ a *diagonal weak homotopy n-equivalences* if the induced map of reduced simplicial sets $d(f) = diag(f)$ defined by $d(f)_k = f_{k,k}$ for every $k \geq 0$, is a weak homotopy n-equivalences.

2.9. Proposition. *For $0 \leq n \leq \infty$, the category $\mathbf{ssSet}_0$ of reduced bisimplicial sets supports a letf proper, combinatorial and simplicial model category structure denoted by $\big(\mathbf{ssSet}_0, d^\star\mathbf{W}_n, \underline{\mathrm{Hom}}^{(1)}\big)$ where the weak equivalences are the diagonal weak homotopy n-equivalences, the cofibrations are the injections and the simplicial enrichment is given by:*

$$\begin{aligned}
\underline{\mathrm{Hom}}^{(1)}_{\mathbf{ssSet}_0}\big(X, Y\big)_m &= \mathrm{Hom}_{\mathbf{ssSet}_0}\Big(\big(X \times \mathrm{p}_1^*(\Delta^m)\big)/\big(\star \times \mathrm{p}_1^*(\Delta^m)\big), Y\Big) \\
&\cong \mathrm{Hom}_{\mathbf{ssSet}}\big(X \times \mathrm{p}_1^*(\Delta^m), Y\big)
\end{aligned} \tag{12}$$

where $\mathrm{p}_1^*(K)_{p,q} = K_p$ for every simplicial set K.

Proof. Since the category $\mathbf{ssSet}_0$ is isomorphic to the category of functors $\big(\mathbf{sSet}_0\big)^{\Delta^{op}}$ we deduce from [Dugger, 2001] and the Proposition 2.1 that there is a left proper, combinatorial and simplicial model category structure on $\mathbf{ssSet}_0$ called *the universal simplicial replacement* of the model category $\big(\mathbf{sSet}_0, \nu^\star\mathbf{W}_n\big)$. We shall prove that this universal simplicial replacement is $\big(\mathbf{ssSet}_0, d^\star\mathbf{W}_n, \underline{\mathrm{Hom}}^{(1)}\big)$.

By definition it is a Bousfield localization of the Reedy model category $\big(\mathbf{ssSet}_0, \mathbf{W}_n^{Rdy}\big)$ whose weak equivalences are the maps $f\colon\ X \longrightarrow Y$ verifying that $f_{p,\bullet}$ is a weak homotopy n-equivalences for every $p \geq 0$, and the cofibrations are the maps $f\colon\ X \longrightarrow Y$ such that for every $p \geq 0$ the dashed arrow in the following pushout diagram of $\mathbf{sSet}_0$:

$$\begin{tikzcd}
\operatorname{colim} X_{m,\bullet} \arrow[r] \arrow[d, "\operatorname{colim} f_{m,\bullet}"'] & X_{p,\bullet} \arrow[d] \arrow[ddr, bend left, "f_{p,\bullet}"] & \\
\operatorname{colim} Y_{m,\bullet} \arrow[r] \arrow[drr, bend right] & \operatorname{colim} Y_{m,\bullet} \coprod\limits_{\operatorname{colim} X_{m,\bullet}} X_{p,\bullet} \arrow[dr, dashed] & \\
 & & Y_{p,\bullet}
\end{tikzcd} \tag{13}$$

is an injection of reduced simplicial sets, where the colimits colim $X_{m,\bullet}$ and colim $Y_{m,\bullet}$ are calculated in the category $\mathbf{sSet}_0$ and defined over the full subcategory of $[p] \big| \Delta$ whose objects are the non-identity surjections $[p] \to [m]$.

It follows in particular that the cofibrations are the injections of reduced bisimplicial sets. Indeed all the colimits in the diagram (13) (including the pushout) can be calculated in the category $\mathbf{sSet}$ because they are all defined over small connected categories.

On the other hand consider the functors:

$$\mathbf{ssSet}_0 \xrightarrow{d} \mathbf{sSet}_0 \ , \quad \mathbf{sSet}_0 \xrightarrow{r} \mathbf{ssSet}_0 \quad \text{and} \quad \mathbf{sSet}_0 \xrightarrow{k} \mathbf{ssSet}_0 \tag{14}$$

where d is the diagonal functor for reduced bisimplicial sets $d(X)_k = X_{k,k}$, the functor r is a right adjoint of d defined for example by:

$$r(A)_{p,q} \ = \ \operatorname{Hom}_{\mathbf{sSet}_0}\big((\Delta^p \times \Delta^q)/(\Delta^p \times \Delta^q_0), A\big) \ = \ \big(A^{\wedge\Delta^p}\big)_q \qquad \text{(see (3))}$$

and k is the constant diagram functor defined by $k(A)_{p,q} = A_q$.

We find that $k\big(\nu^\star\mathbf{W}_n\big) \subseteq \mathbf{W}_n^{Red}$, the functor d preserves the injections and that $d\big(\mathbf{W}_n^{Rdy}\big) \subseteq \nu^\star\mathbf{W}_n$ (see the Corollary 2.3.17 and the Theorem 1.4.3 from [Cisinski, 2006]). In particular $d \dashv r$ is a Quillen adjuntion with respect to the model categories $\big(\mathbf{ssSet}_0, \mathbf{W}_n^{Rdy}\big)$ and $\big(\mathbf{sSet}_0, \nu^\star\mathbf{W}_n\big)$ and there exist functors $\widetilde{k}$ and $\widetilde{d}$ making commutative the following diagrams:

$$\begin{tikzcd}
\mathbf{sSet}_0 \arrow[r, "k"] \arrow[d] & \mathbf{ssSet}_0 \arrow[d] & & \mathbf{ssSet}_0 \arrow[r, "d"] \arrow[d] & \mathbf{sSet}_0 \arrow[d] \\
\mathbf{sSet}_0\Big[\big(\nu^\star\mathbf{W}_n\big)^{-1}\Big] \arrow[r, dashed, "\widetilde{k}"'] & \mathbf{ssSet}_0\Big[\big(\mathbf{W}_n^{Rdy}\big)^{-1}\Big] & \text{and} & \mathbf{ssSet}_0\Big[\big(\mathbf{W}_n^{Rdy}\big)^{-1}\Big] \arrow[r, dashed, "\widetilde{d}"'] & \mathbf{sSet}_0\Big[\big(\nu^\star\mathbf{W}_n\big)^{-1}\Big] .
\end{tikzcd}$$

Furthermore we have a natural transformation $\eta\colon\ r \Longrightarrow k\ :\ \mathbf{sSet}_0 \longrightarrow \mathbf{ssSet}_0$ defined for a reduced simplicial set A and $p \geq 0$ as the map:

$$\big(\eta_A\big)_{p,\bullet}\colon\ r(A)_{p,\bullet} \ = \ A^{\wedge\Delta^p} \xrightarrow{A^{\wedge\delta}} A^{\wedge\Delta^0} \ \cong \ k(A)_{p,\bullet} \quad \text{where} \quad \delta = \underbrace{\delta_0 \circ \cdots \circ \delta_0}_{p}\colon\ \Delta^0 \longrightarrow \Delta^p \ .$$

It follows in particular that η_A is an element of $\mathbf{W}_n^{Rdy}$ if the reduced simplicial set A is a Kan complex. Hence we have an adjunction:

$$\begin{tikzcd}\mathbf{ssSet}_0\big[\big(\mathbf{W}_n^{Rdy}\big)^{-1}\big] \arrow[r, bend left, "\widetilde{d}"] \arrow[r, phantom, "\perp"] & \mathbf{sSet}_0\big[\big(\nu^\star \mathbf{W}_n\big)^{-1}\big] \arrow[l, bend left, "\widetilde{k}"]\end{tikzcd},$$

that is to say $\widetilde{d}$ is a homotopy colimit functor in the model category $\big(\mathbf{sSet}_0, \nu^\star \mathbf{W}_n\big)$ of diagrams over Δ^{op}.

Finally notice that the canonical simplicial enrichment of the category $\mathbf{ssSet}_0 \cong \big(\mathbf{sSet}_0\big)^{\Delta^{op}}$ of simplicial objects of $\mathbf{sSet}_0$ is defined by $\underline{\mathrm{Hom}}(X,Y)_m = \mathrm{Hom}_{\mathbf{ssSet}_0}\big(X \otimes \Delta^m, Y\big)$ where:

$$\big(X \otimes \Delta^m\big)_{p,q} \;=\; \Big(\coprod_{\Delta^m_p} X_{p,q}\Big)\Big/\Big(\coprod_{\Delta^m_p} \star\Big) \;\cong\; \Big(X_{p,q} \times \Delta^m_p\Big)\Big/\Big(\star \times \Delta^m_p\Big)\,.$$

So $\big(\mathbf{ssSet}_0, d^\star \mathbf{W}_n, \underline{\mathrm{Hom}}^{(1)}\big)$ is the universal simplicial replacement of the model category $\big(\mathbf{sSet}_0, \nu^\star \mathbf{W}_n\big)$. ■

For the next statement recalls that for every K and L simplicial sets we denote $K \boxtimes L$ the bisimplicial set $\mathrm{p}_1^* K \times \mathrm{p}_2^* L$ where $\mathrm{p}_1^*(K)_{p,q} = K_p$ and $\mathrm{p}_2^*(L)_{p,q} = L_q$.

2.10. Corollary. *Let $0 \leq n \leq \infty$. A reduced bisimplicial set X is a fibrant object of the model category $\big(\mathbf{ssSet}_0, d^\star \mathbf{W}_n\big)$ of the Proposition 2.9 if and only if for every map $\varphi\colon\ [p] \longrightarrow [p']$ of the simplex category Δ the induced map $\varphi^*\colon\ X_{p',\bullet} \longrightarrow X_{p,\bullet}$ is a weak homotopy n-equivalence of reduced simplicial sets and for every $p \geq 0$ the map:*

$$\underline{\mathrm{Hom}}^{(2)}_{\mathbf{ssSet}}\big(\mathrm{p}_1^*\Delta^p, X\big) \longrightarrow \underline{\mathrm{Hom}}^{(2)}_{\mathbf{ssSet}}\big(\mathrm{p}_1^*\partial\Delta^p, X\big) \tag{15}$$

is a fibration of the model category $\big(\mathbf{sSet}_0, \nu^\star \mathbf{W}_n\big)$, where:

$$\underline{\mathrm{Hom}}^{(2)}_{\mathbf{ssSet}}\big(Z, X\big)_q :\; = \mathrm{Hom}_{\mathbf{ssSet}}\big(Z \times \mathrm{p}_2^*(\Delta^q), X\big)\,.$$

In particular X is a fibrant object of the model category $\big(\mathbf{ssSet}_0, d^\star \mathbf{W}_n\big)$ if it satisfies the following properties:

1. *For every map $\varphi\colon\ [p] \longrightarrow [p']$ of the simplex category Δ the induced map of reduced simplicial sets $\varphi^*\colon\ X_{p',\bullet} \longrightarrow X_{p,\bullet}$ is a weak homotopy n-equivalence.*
2. *For $p \geq 0$, $q \geq 2$ and $0 \leq k \leq q$ the function:*

$$\mathrm{Hom}_{\mathbf{ssSet}}\big(\Delta^p \boxtimes \Delta^q, X\big) \longrightarrow \mathrm{Hom}_{\mathbf{ssSet}}\big(\Delta^p \boxtimes \Lambda^{q,k}, X\big) \tag{16}$$

is surjective if $2 \leq q \leq n$ and bijective if $q \geq n+1$.

3. *For $p \geq n+1$ and $q \geq 1$ the function:*

$$\mathrm{Hom}_{\mathbf{ssSet}}\big(\Delta^p \boxtimes \Delta^q, X\big) \longrightarrow \mathrm{Hom}_{\mathbf{ssSet}}\big(\partial\Delta^p \boxtimes \Delta^q, X\big) \tag{17}$$

   is bijective.

4. *For $2 \leq p \leq n$, $2 \leq q \leq n$ and $0 \leq k \leq q$ the function:*

$$\mathrm{Hom}_{\mathbf{ssSet}}\big(\partial\Delta^p \boxtimes \Delta^q, X\big) \longrightarrow \mathrm{Hom}_{\mathbf{ssSet}}\big(\partial\Delta^p \boxtimes \Lambda^{q,k}, X\big) \tag{18}$$

   is surjective.

5. *For $1 \leq p \leq n$, $2 \leq q \leq n$ and $0 \leq k \leq q$ the function:*

$$\mathrm{Hom}_{\mathbf{ssSet}}\Big(\Delta^p \boxtimes \Delta^q, X\Big) \longrightarrow \mathrm{Hom}_{\mathbf{ssSet}}\Big(\partial\Delta^p \boxtimes \Delta^q, X\Big) \underset{\mathrm{Hom}\left(\partial\Delta^p\boxtimes\Lambda^{q,k}, X\right)}{\times} \mathrm{Hom}_{\mathbf{ssSet}}\Big(\Delta^p \boxtimes \Lambda^{q,k}, X\Big) \tag{19}$$

   is surjective.

Proof. For the first part recall from [Dugger, 2001] that a reduced bisimplicial set X is a fibrant object of the model category $\big(\mathbf{ssSet}_0, d^\star \mathbf{W}_n\big)$ if and only if for every map $\varphi\colon\ [p] \longrightarrow [p']$ of the simplex category Δ the induced map $\varphi^*\colon\ X_{p',\bullet} \longrightarrow X_{p,\bullet}$ is a weak homotopy n-equivalence of reduced simplicial sets and X is a fibrant object of the model category $\big(\mathbf{ssSet}_0, \mathbf{W}_n^{Rdy}\big)$.

On the other hand a reduced bisimplicial set X is a fibrant object in $\big(\mathbf{ssSet}_0, \mathbf{W}_n^{Rdy}\big)$ if and only if for every $p \geq 0$ the following map is a fibration $\big(\mathbf{sSet}_0, \nu^\star \mathbf{W}_n\big)$:

$$X_{p,\bullet} \longrightarrow \lim_{\substack{[m]\to[p] \\ \text{inj non id}}} X_{m,\bullet}\,. \tag{20}$$

From the isomorphisms:

$$\big(\lim X_{m,\bullet}\big)_q \;\cong\; \lim X_{m,q} \;\cong\; \underline{\mathrm{Hom}}_{\mathbf{ssSet}}\Big(\big(\lim \Delta^m\big) \boxtimes \Delta^q, X\Big) \;\cong\; \underline{\mathrm{Hom}}_{\mathbf{ssSet}}\big(\partial\Delta^p \boxtimes \Delta^q, X\big)\,,$$

we deduce that X is a fibrant object of $\big(\mathbf{ssSet}_0, \mathbf{W}_n^{Rdy}\big)$ if and only if for every $p \geq 0$ the map (15) is a fibration of $\big(\mathbf{sSet}_0, \nu^\star \mathbf{W}_n\big)$.

For the second part note that conditions 2-5 imply that (15) is a map between fibrant objects of the model category $\big(\mathbf{sSet}_0, \nu^\star \mathbf{W}_n\big)$ of the Proposition 2.1 verifying the right lifting property with respect to the maps of the set J_0 of (10) for every $p \geq 0$. Then (15) is a fibration of $\big(\mathbf{sSet}_0, \nu^\star \mathbf{W}_n\big)$ according to Proposition 2.5. ■

It is not difficult to show that for $0 \leq n \leq \infty$ the adjunctions:

$$\begin{tikzcd}
\mathbf{ssSet}_0 \arrow[r, bend right, "(\,\cdot\,)_{0,\bullet}"'] \arrow[r, phantom, "\perp"] & \mathbf{sSet}_0 \arrow[l, bend right, "k"'] & \text{and} & \mathbf{ssSet}_0 \arrow[r, bend left, "d"] \arrow[r, phantom, "\perp"] & \mathbf{sSet}_0 \arrow[l, bend left, "r"]
\end{tikzcd} \tag{21}$$

where d, r and k are the functors (14) are Quillen equivalences between the model category of the Propositions 2.9 and 2.1 (see also the adjuntion (9)).

3. The geometric nerve for 2-groups

Recall that a *monoidal category* (see for exemple [Borceux, 1994] or [Joyal, Street, 1993]) is a category $\mathcal{M}$ equipped with a functor $\otimes\colon\ \mathcal{M}\times\mathcal{M} \longrightarrow \mathcal{M}$, a distinguished object $\mathbb{1}$ and natural isomorphisms:

$$\Big\{ (X{\otimes}Y){\otimes}Z \xrightarrow{a_{X,Y,Z}} X{\otimes}(Y{\otimes}Z) \Big\}_{X,Y,Z}, \qquad \Big\{ X \xrightarrow{l_X} \mathbb{1}{\otimes}X \Big\}_X \quad \text{and} \quad \Big\{ X \xrightarrow{r_X} X{\otimes}\mathbb{1} \Big\}_X ;$$

such that the following diagrams are commutative:

$$\begin{tikzcd}
\big((W{\otimes}X){\otimes}Y\big){\otimes}Z \arrow[r, "a"] \arrow[d, "a\otimes Z"'] & (W{\otimes}X){\otimes}(Y{\otimes}Z) \arrow[r, "a"] & W{\otimes}\big(X{\otimes}(Y{\otimes}Z)\big) \\
\big(W{\otimes}(X{\otimes}Y)\big){\otimes}Z \arrow[rr, "a"'] & & W{\otimes}\big((X{\otimes}Y){\otimes}Z\big) \arrow[u, "W\otimes a"']
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
(X{\otimes}\mathbb{1}){\otimes}Y \arrow[rr, "a_{X,\mathbb{1},Y}"] & & X{\otimes}(\mathbb{1}{\otimes}Y) \\
& X{\otimes}Y \arrow[ul, "r_X\otimes Y"] \arrow[ur, "X\otimes l_Y"'] &
\end{tikzcd} \tag{22}$$

for every objects X, Y, Z and W of $\mathcal{M}$.

If A and B are objects of a monoidal category it follows that the triangles:

$$\begin{tikzcd}
(A{\otimes}B){\otimes}\mathbb{1} \arrow[rr, "a_{A,B,\mathbb{1}}"] & & A{\otimes}(B{\otimes}\mathbb{1}) \\
& A{\otimes}B \arrow[ul, "r_{A\otimes B}"] \arrow[ur, "A\otimes r_B"'] &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
& A{\otimes}B \arrow[dl, "l_A\otimes B"'] \arrow[dr, "l_{A\otimes B}"] & \\
(\mathbb{1}{\otimes}A){\otimes}B \arrow[rr, "a_{\mathbb{1},A,B}"'] & & \mathbb{1}{\otimes}(A{\otimes}B)
\end{tikzcd}$$

are commutative. Even more we can show that $l_{\mathbb{1}} = r_{\mathbb{1}}$ (see [Joyal, Street, 1993] or [Kelly, 1964]).

A *normal lax monoidal functor* of monoidal categories is a functor $F\colon\ \mathcal{M} \longrightarrow \mathcal{N}$ satisfying $F\big(\mathbb{1}_{\mathcal{M}}\big) = \mathbb{1}_{\mathcal{N}}$ equipped with a natural transformation:

$$\Big\{ F(X)\otimes F(Y) \xrightarrow{m^F_{X,Y}} F(X\otimes Y) \Big\}_{X,Y} \tag{23}$$

such that the following diagrams are commutative:

$$\begin{tikzcd}
F\big((X{\otimes}Y){\otimes}Z\big) \arrow[d, "Fa"'] & F(X{\otimes}Y){\otimes}FZ \arrow[l, "m^F"'] & (FX{\otimes}FY){\otimes}FZ \arrow[l, "m^F\otimes FZ"'] \arrow[d, "a"] \\
F\big(X{\otimes}(Y{\otimes}Z)\big) & FX{\otimes}F(Y{\otimes}Z) \arrow[l, "m^F"] & FX{\otimes}(FY{\otimes}FZ) \arrow[l, "FX\otimes m^F"]
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathbb{1}{\otimes}FX \arrow[d, "m^F_{\mathbb{1},X}"'] & & FX{\otimes}\mathbb{1} \arrow[d, "m^F_{X,\mathbb{1}}"] \\
& FX \arrow[ul, "l_{FX}"'] \arrow[ur, "r_{FX}"] \arrow[dl, "Fl_X"] \arrow[dr, "Fr_X"'] & \\
F(\mathbb{1}{\otimes}X) & & F(X{\otimes}\mathbb{1})
\end{tikzcd} \tag{24}$$

for all objects X, Y and Z of $\mathcal{M}$.

The small monoidal categories and the normal lax monoidal functors form in a canonical way a pointed category $\mathbf{cat}^{\otimes}_{Nlax}$. There is a (strict) 2-category $\underline{\mathbf{cat}}^{\otimes}_{Nlax}$ whose underlying category is $\mathbf{cat}^{\otimes}_{Nlax}$ and the 2-arrows are the *monoidal natural transformations* of normal lax monoidal functors, defined as the natural transformations $\eta\colon F \Longrightarrow G$ between the underlying functors $F, G\colon \mathcal{M} \longrightarrow \mathcal{N}$ such that $\eta_{\mathbb{1}_{\mathcal{M}}} = \mathrm{id}_{\mathbb{1}_{\mathcal{N}}}$ and that the following diagram is commutative:

$$\begin{tikzcd}
FX \otimes FY \arrow[r, "\eta_X \otimes \eta_Y"] \arrow[d, "m^F_{X,Y}"'] & GX \otimes GY \arrow[d, "m^G_{X,Y}"] \\
F(X \otimes Y) \arrow[r, "\eta_{(X\otimes Y)}"'] & G(X \otimes Y)
\end{tikzcd} \tag{25}$$

for all objects X and Y of $\mathcal{M}$.

It is not difficult to show that the forgetful 2-functor $\underline{\mathbf{cat}}^{\otimes}_{Nlax} \longrightarrow \underline{\mathbf{cat}}$ to the 2-category of small categories, functors and natural transformations reflects the internal equivalences and the isomorphisms.

3.1. We call an object X in a monoidal category $\mathcal{M}$ *invertible* if it verifies one of the following equivalent conditions:

1. The functors $X \otimes \bullet\colon \mathcal{M} \longrightarrow \mathcal{M}$ and $\bullet \otimes X\colon \mathcal{M} \longrightarrow \mathcal{M}$ are equivalences of categories.
2. There exist objects X' and X'' and isomorphisms $X' \otimes X \xrightarrow[\cong]{\alpha_X} \mathbb{1} \xleftarrow[\cong]{\beta_X} X \otimes X''$ of $\mathcal{M}$.
3. There exist an object X' and isomorphisms $X' \otimes X \xrightarrow[\cong]{\alpha_X} \mathbb{1} \xleftarrow[\cong]{\beta_X} X \otimes X'$ of $\mathcal{M}$.

A 2-*group* is a monoidal category whose underlying category is a groupoid and in which all objects are invertible.

3.2. Lemma. *Let $\mathcal{M}$ be a monoidal category whose underlying category is a groupoid, then $\mathcal{M}$ is a 2-group if and only if there exist a functor $\iota\colon \mathcal{M} \longrightarrow \mathcal{M}$ and natural isomorphisms:*

$$\left\{ \iota(X) \otimes X \xrightarrow[\cong]{\alpha_X} \mathbb{1} \xleftarrow[\cong]{\beta_X} X \otimes \iota(X) \right\}_X .$$

Proof. If $\mathcal{M}$ is a monoidal category whose underlying category is a groupoid, then it is not difficult to prove (see for exemple [Baez, Lauda, 2004]) that $\mathcal{M}$ is a 2-group if and only if there exist functors $\iota^l\colon \mathcal{M} \longrightarrow \mathcal{M}$ and $\iota^r\colon \mathcal{M} \longrightarrow \mathcal{M}$ and natural isomorphisms:

$$\left\{ \iota^l(X) \otimes X \xrightarrow[\cong]{\overline{\alpha}_X} \mathbb{1} \xleftarrow[\cong]{\overline{\beta}_X} X \otimes \iota^r(X) \right\}_X .$$

In this case note that there is a natural isomorphism η between the functors ι^l and ι^r. Indeed for any object X of $\mathcal{M}$ we have:

$$\begin{tikzcd}
\iota^l(X) \arrow[rr, dashed, "\eta_X"] \arrow[d, "r"'] & & \iota^r(X) \\
\iota^l(X)\otimes \mathbb{1} \arrow[d, "\iota^l(X)\otimes\overline{\beta}_X^{-1}"'] & & \mathbb{1}\otimes\iota^r(X) \arrow[u, "l"'] \\
\iota^l(X)\otimes\big(X\otimes\iota^r(X)\big) \arrow[rr, "a"'] & & \big(\iota^l(X)\otimes X\big)\otimes\iota^r(X) \arrow[u, "\overline{\alpha}_X\otimes\iota^r(X)"']
\end{tikzcd}$$

Then take $\iota = \iota^l$, $\alpha_X = \overline{\alpha}_X$ and $\beta_X = \overline{\beta}_X \circ (X \otimes \eta)$. ∎

Denote by 2-**Grp** the category of small 2-groups $\big($resp. 2-$\underline{\mathbf{Grp}}$ the 2-category of small 2-groups$\big)$, namely the full subcategory of $\mathbf{cat}^{\otimes}_{Nlax}$ $\big($resp. the full 2-subcategory of $\underline{\mathbf{cat}}^{\otimes}_{Nlax}\big)$ whose objects are the small 2-groups. The *homotopy category of* 2*-groups* denoted by 2-h**Grp** is the category where the objects are the small 2-groups and the morphisms are the isomorphism classes of objects of the groupoid of morphisms in the 2-category 2-$\underline{\mathbf{Grp}}$.

If $\mathcal{G}$ is a small 2-group then we denote by $\pi_0(\mathcal{G})$ the set of path components of $\mathcal{G}$ namely the set of isomorphism classes of objects of $\mathcal{G}$. The functor $\otimes$ induces on the set $\pi_0(\mathcal{G})$ a group structure were the unit is the class of the object $\mathbb{1}$. We call $\pi_0(\mathcal{G})$ the *group of path components of* $\mathcal{G}$. We also denote by $\pi_1(\mathcal{G})$ the group of automorphisms of the object $\mathbb{1}$ in the category $\mathcal{G}$, which is always a commutative group, and we call it the *fundamental group of* $\mathcal{G}$. We have functors $\pi_0, \pi_1\colon$ 2-$\mathbf{Grp} \longrightarrow \mathbf{Grp}$.

A morphism of 2-groups $F\colon\ \mathcal{G} \longrightarrow \mathcal{G}'$ is called a *weak equivalence* if it satisfies one of the following equivalent conditions:

1. $\pi_0 F$ and $\pi_1 F$ are isomorphisms of groups.
2. The underlying functor of F is an equivalence of small categories.
3. F is an internal equivalence in the 2-category 2-$\underline{\mathbf{Grp}}$.
4. The image of F by the canonical functor 2-$\mathbf{Grp} \longrightarrow$ 2-$h\mathbf{Grp}$ is an isomorphism.

3.3. Remark. The 2-categories $\underline{\mathbf{cat}}^{\otimes}_{Nlax}$ and 2-$\underline{\mathbf{Grp}}$ are both cotensored 2-categories. Namely there are 2-functors:

$$\begin{tikzcd}
\underline{\mathbf{cat}}^{op}\times\underline{\mathbf{cat}}^{\otimes}_{Nlax} \arrow[r] & \underline{\mathbf{cat}}^{\otimes}_{Nlax} \\
(\mathcal{A},\mathcal{M}) \arrow[r, mapsto] & \mathcal{M}^{\mathcal{A}}
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
\underline{\mathbf{cat}}^{op}\times 2\text{-}\underline{\mathbf{Grp}} \arrow[r] & 2\text{-}\underline{\mathbf{Grp}} \\
(\mathcal{A},\mathcal{G}) \arrow[r, mapsto] & \mathcal{G}^{\mathcal{A}}
\end{tikzcd} \tag{26}$$

inducing 2-adjuntions:

$$\begin{tikzcd}
\big(\underline{\mathbf{cat}}^{\otimes}_{Nlax}\big)^{op} \arrow[r, bend right, "\underline{\mathrm{Hom}}_{\mathbf{cat}^{\otimes}_{Nlax}}(\,\cdot\,{,}\mathcal{M})"'] \arrow[r, phantom, "\perp"] & \underline{\mathbf{cat}} \arrow[l, bend right, "\mathcal{M}^{\cdot}"']
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
2\text{-}\underline{\mathbf{Grp}}^{op} \arrow[r, bend right, "\underline{\mathrm{Hom}}_{2\text{-}\mathbf{Grp}}(\,\cdot\,{,}\mathcal{G})"'] \arrow[r, phantom, "\perp"] & \underline{\mathbf{cat}} \arrow[l, bend right, "\mathcal{G}^{\cdot}"']
\end{tikzcd}$$

for every monoidal category $\mathcal{M}$ and every 2-group $\mathcal{G}$.

Indeed let $\mathcal{A}$ be a small category and $\mathcal{M}$ be a monoidal category. The underlying category of the monoidal category $\mathcal{M}^{\mathcal{A}}$ is by definition the category of functors and natural transformations from $\mathcal{A}$ to the underlying category of the monoidal category $\mathcal{M}$. On the other hand using the 2-functor $(\,\cdot\,)^{\mathcal{A}}\colon\ \underline{\mathbf{cat}} \longrightarrow \underline{\mathbf{cat}}$ we can pass the monoidal structure of $\mathcal{M}$ into $\mathcal{M}^{\mathcal{A}}$.

Explicitly if $\mathcal{X}, \mathcal{Y}\colon\ \mathcal{A} \longrightarrow \mathcal{M}$ are objects of $\mathcal{M}^{\mathcal{A}}$ the product $\mathcal{X} \otimes \mathcal{Y}\colon\ \mathcal{A} \longrightarrow \mathcal{M}$ is the functor defined in an object A of $\mathcal{A}$ as $\mathcal{X}_A \otimes \mathcal{Y}_A$ and in a morphism f of $\mathcal{A}$ as $\mathcal{X}_f \otimes \mathcal{Y}_f$. The distinguished object $\mathbb{1}$ of $\mathcal{M}^{\mathcal{A}}$ is equal to the constant functor with value the distinguished object $\mathbb{1}$ of $\mathcal{M}$. The natural isomorphisms:

$$\Big\{ (\mathcal{X}\otimes\mathcal{Y})\otimes\mathcal{Z} \xrightarrow{a_{\mathcal{X},\mathcal{Y},\mathcal{Z}}} \mathcal{X}\otimes(\mathcal{Y}\otimes\mathcal{Z}) \Big\}_{\mathcal{X},\mathcal{Y},\mathcal{Z}}, \qquad \Big\{ \mathcal{X} \xrightarrow{l_{\mathcal{X}}} \mathbb{1}\otimes\mathcal{X} \Big\}_{\mathcal{X}} \quad \text{and} \quad \Big\{ \mathcal{X} \xrightarrow{r_{\mathcal{X}}} \mathcal{X}\otimes\mathbb{1} \Big\}_{\mathcal{X}}$$

are defined for any objects $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ of $\mathcal{M}^{\mathcal{A}}$ and any object A of $\mathcal{A}$ as the morphisms of $\mathcal{M}$:

$$\big(\mathcal{X}_A\otimes\mathcal{Y}_A\big)\otimes\mathcal{Z}_A \xrightarrow{a_{\mathcal{X}_A,\mathcal{Y}_A,\mathcal{Z}_A}} \mathcal{X}_A\otimes\big(\mathcal{Y}_A\otimes\mathcal{Z}_A\big)\,, \qquad \mathcal{X}_A \xrightarrow{l_{\mathcal{X}_A}} \mathbb{1}\otimes\mathcal{X}_A \quad \text{and} \quad \mathcal{X}_A \xrightarrow{r_{\mathcal{X}_A}} \mathcal{X}_A\otimes\mathbb{1}\ .$$

Notice that if $\mathcal{M} = \mathcal{G}$ is a 2-group then the monoidal category just defined $\mathcal{G}^{\mathcal{A}}$ is also a 2-group because the category of functors $\mathcal{G}^{\mathcal{A}}$ is a groupoid and by the Lemma 3.2.

Let is consider $\varphi\colon\ \mathcal{B} \longrightarrow \mathcal{A}$ and $(F, m^F)\colon\ \mathcal{M} \longrightarrow \mathcal{N}$ a functor and a normal lax monoidal functor respectively. The normal lax monoidal functor $(F^{\varphi}, m^{F^{\varphi}})\colon\ \mathcal{M}^{\mathcal{A}} \longrightarrow \mathcal{N}^{\mathcal{B}}$ is given by the functor $F^{\varphi} = F \circ - \circ \varphi$ and the natural transformation:

$$\Big\{ F^{\varphi}(\mathcal{X}) \otimes F^{\varphi}(\mathcal{Y}) \xrightarrow{m^{F^{\varphi}}_{\mathcal{X},\mathcal{Y}}} F^{\varphi}(\mathcal{X}\otimes\mathcal{Y}) \Big\}_{\mathcal{X},\mathcal{Y}}$$

defined in the objects $\mathcal{X}$ and $\mathcal{Y}$ of $\mathcal{M}^{\mathcal{A}}$ and the object B of $\mathcal{B}$ as the morphism:

$$F(\mathcal{X}_{\varphi B}) \otimes F(\mathcal{Y}_{\varphi B}) \xrightarrow{m^{F}_{\mathcal{X}_{\varphi B},\mathcal{Y}_{\varphi B}}} F(\mathcal{X}_{\varphi B} \otimes \mathcal{Y}_{\varphi B})\ .$$

Finally let $\alpha\colon\ F \Rightarrow G\ :\ \mathcal{B} \rightarrow \mathcal{A}$ and $\eta\colon\ (\varphi, m^{\varphi}) \Rightarrow (\psi, m^{\psi})\ :\ \mathcal{M} \rightarrow \mathcal{N}$ be a natural transformation of functors and a monoidal natural transformation of normal lax monoidal functors respectively. We define $\eta^{\alpha}\colon\ (F^{\varphi}, m^{F^{\varphi}}) \Rightarrow (G^{\psi}, m^{G^{\psi}})$ as the the monoidal natural transformation $\mathcal{N}^{\alpha} \star \eta^{\mathcal{A}} = \eta^{\mathcal{B}} \star \mathcal{M}^{\alpha}$.

It is routine demonstrate that for any small category $\mathcal{A}$ and any small monoidal categories $\mathcal{M}$ and $\mathcal{N}$ there is a natural isomorphism of categories:

$$\underline{\mathrm{Hom}}_{\mathbf{cat}}\Big(\mathcal{A}, \underline{\mathrm{Hom}}_{\mathbf{cat}^{\otimes}_{Nlax}}(\mathcal{N},\mathcal{M})\Big) \cong \underline{\mathrm{Hom}}_{\mathbf{cat}^{\otimes}_{Nlax}}(\mathcal{N},\mathcal{M}^{\mathcal{A}})\,.$$

3.3.1. Denote by $\underline{\Delta}$ the full 2-subcategory of $\underline{\mathbf{cat}}$ whose objects are the totally ordered sets $[n] = \{0 < \cdots < n\}$ for $n \geq 0$. In other words $\underline{\Delta}$ is the 2-category whose underlying category is the simplex category Δ and where there exist a 2-arrow $\begin{tikzcd} {[n]} \arrow[r, bend left, "\varphi"{name=U}] \arrow[r, bend right, "\psi"'{name=D}] & {[m]} \arrow[Rightarrow, from=U, to=D] \end{tikzcd}$ if and only if $\varphi(i) \leq \psi(i)$ for all $0 \leq i \leq n$.

If $\mathcal{G}$ is a 2-group, consider the 2-functor $\mathcal{G}^{\bullet}\colon\ \underline{\Delta}^{op} \longrightarrow 2\text{-}\underline{\mathbf{Grp}}$ induced by the exponentiation (26).

3.4. Lemma. *Let $\mathcal{G}$ be a 2-group and $\varphi\colon\ [n] \longrightarrow [m]$ be a morphism of the simplex category Δ. Then the induced morphism $\varphi^{\star}\colon\ \mathcal{G}^{[m]} \longrightarrow \mathcal{G}^{[n]}$ is a weak equivalence of 2-groups.*

Proof. Every morphism $\varphi\colon\ [n] \longrightarrow [m]$ is the simplex category Δ is equal to a composition of the form $\varphi = \delta_{j_{m-n+k}} \circ \cdots \circ \delta_{j_1} \circ \sigma_{i_k} \circ \cdots \circ \sigma_{i_1}$ where δ_j and σ_i are the faces and degeneracy maps. So we just need to verify the statement for the maps δ_j and σ_i for all $0 \leq i \leq k$ and $0 \leq j \leq k+1$.

Notice that for $0 \leq i \leq k$ we have the equality $\sigma_i \circ \delta_i = \mathrm{id}_{[k]}$. On the other hand, there is a 2-arrow $\mathrm{id}_{[k+1]} \Rightarrow \delta_i \circ \sigma_i$ of the 2-category $\underline{\Delta}$ for $0 \leq i \leq k$, because $a \leq \delta_i \circ \sigma_i(a)$ if $0 \leq a \leq n+1$. Then the induced morphisms $\delta_j^{\star}$ and $\sigma_i^{\star}$ are internal equivalences in $2\text{-}\underline{\mathbf{Grp}}$ for $0 \leq i, j \leq k$.

Finally to show that $\delta_{k+1}^{\star}\colon\ \mathcal{G}^{[k+1]} \longrightarrow \mathcal{G}^{[k]}$ is an internal category in $2\text{-}\underline{\mathbf{Grp}}$, notice that $\sigma_k \circ \delta_{k+1} = \mathrm{id}_{[k]}$ and that there exists a 2-arrow $\delta_{k+1} \circ \sigma_k \Rightarrow \mathrm{id}_{[k+1]}$ of the 2-category $\underline{\Delta}$. ■

3.5. Let $\mathcal{M}$ be a monoidal category and $q \geq 0$ be a natural number. A *q-simplex* of $\mathcal{M}$ is a pair (X, α) where:

$$X = \Big\{ X_{ij} \,\Big|\, 0 \leq i < j \leq q \Big\} \quad \text{and} \quad \alpha = \Big\{ \alpha_{ijk}\colon\ X_{ij} \otimes X_{jk} \longrightarrow X_{ik} \,\Big|\, 0 \leq i < j < k \leq q \Big\}$$

are families of objects and morphisms of $\mathcal{M}$ respectively, such that:

$$\begin{tikzcd}
X_{il} & X_{ij} \otimes X_{jl} \arrow[l, "\alpha_{ijl}"'] \\
 & X_{ij} \otimes \big(X_{jk} \otimes X_{kl}\big) \arrow[u, "X_{ij} \otimes \alpha_{jkl}"'] \\
X_{ik} \otimes X_{kl} \arrow[uu, "\alpha_{ikl}"] & \big(X_{ij} \otimes X_{jk}\big) \otimes X_{kl} \arrow[l, "\alpha_{ijk} \otimes X_{kl}"] \arrow[u, "a"', "\wr"]
\end{tikzcd}$$

is a commutative diagram for every $0 \leq i < j < k < l \leq q$.

Let $\mathcal{M}_q$ denote the set of q-simplices of $\mathcal{M}$, there is a functor:

$$\begin{tikzcd}
\mathbf{cat}^{\otimes}_{Nlax} \times \Delta^{op} \arrow[r] & \mathbf{Set} \\
\big(\mathcal{M}, [q]\big) \arrow[r, mapsto] & \mathcal{M}_q
\end{tikzcd} \tag{27}$$

defined as follows: If $\varphi\colon\ [q] \longrightarrow [q']$ is a morphism of Δ then the function $\varphi^*\colon\ \mathcal{M}_{q'} \longrightarrow \mathcal{M}_q$ is defined as $\varphi^*(X,\alpha) = (Y,\beta)$ where:

$$Y_{ij} \ = \begin{cases} \mathbb{1}_{\mathcal{G}} & \text{if} \quad \varphi i = \varphi j \\ X_{\varphi i \varphi j} & \text{if} \quad \varphi i < \varphi j \end{cases} \qquad \text{for } 0 \leq i < j \leq q$$

$$\text{and} \qquad \beta_{ijk} \ = \begin{cases} l^{-1}_{X_{\varphi i\varphi k}}\colon\ \mathbb{1} \otimes X_{\varphi i \varphi k} \longrightarrow X_{\varphi i \varphi k} & \text{if} \quad \varphi i = \varphi j \leq \varphi k \\ r^{-1}_{X_{\varphi i\varphi k}}\colon\ X_{\varphi i \varphi k} \otimes \mathbb{1} \longrightarrow X_{\varphi i \varphi k} & \text{if} \quad \varphi i \leq \varphi j = \varphi k \\ \alpha_{\varphi i \varphi j \varphi k}\colon\ X_{\varphi i \varphi j} \otimes X_{\varphi j \varphi k} \longrightarrow X_{\varphi i \varphi k} & \text{if} \quad \varphi i < \varphi j < \varphi k \end{cases}$$

for $0 \leq i < j < k \leq q$.

If $(F, m^F)\colon\ \mathcal{M} \longrightarrow \mathcal{M}'$ is a normal lax monoidal functor of monoidal categories and $q \geq 0$, then the function $(F,m^F)_q\colon\ \mathcal{M}_q \longrightarrow \mathcal{M}'_q$ is defined by $(F,m^F)_q(X,\alpha) = (Y,\beta)$ where $Y_{ij} = F(X_{ij})$ if $0 \leq i < j \leq q$ and:

$$\beta_{ijk} \ = \ \Big(Y_{ij} \otimes Y_{jk} = F(X_{ij}) \otimes F(X_{jk}) \xrightarrow{m^F_{X_{ij},X_{jk}}} F(X_{ij} \otimes X_{jk}) \xrightarrow{F(\alpha_{ijk})} F(X_{ik}) = Y_{ik} \Big)$$

if $0 \leq i < j < k \leq q$.

By adjunction we deduce from the functor (27) the *geometric nerve functor for monoidal categories*:

$$\mathbf{cat}^{\otimes}_{Nlax} \xrightarrow{\mathcal{N}} \mathbf{sSet}_0 \qquad \text{where} \qquad \mathcal{N}\big(\mathcal{M}\big)_q \ = \ \mathcal{M}_q\,. \tag{28}$$

Explicitly for a monoidal category $\mathcal{M}$ the reduced simplicial set $\mathcal{N}\big(\mathcal{M}\big)$ has one 1-simplex for each object of $\mathcal{M}$. The set $\mathcal{N}\big(\mathcal{M}\big)_2$ is identified with the set of morphisms of $\mathcal{M}$ of the form $\alpha_{012}\colon\ X_{01} \otimes X_{12} \longrightarrow X_{02}$ and $\mathcal{N}\big(\mathcal{M}\big)_3$ with the set of commutative diagrams of $\mathcal{M}$ of the form:

$$\begin{tikzcd}
X_{03} & X_{01} \otimes X_{13} \arrow[l, "\alpha_{013}"'] \\
 & X_{01} \otimes \big(X_{12} \otimes X_{23}\big) \arrow[u, "X_{01}\otimes\alpha_{123}"'] \\
X_{02} \otimes X_{23} \arrow[uu, "\alpha_{023}"] & \big(X_{01} \otimes X_{12}\big) \otimes X_{23}\,. \arrow[l, "\alpha_{012}\otimes X_{23}"] \arrow[u, phantom, "a\wr\|"]
\end{tikzcd} \tag{29}$$

Additionally the i-face of the 3-simplex (29) is the 2-simplex α_{jkl} where $0 \leq j < k < l \leq 3$ are different to i, and the i-face of the 2-simplex $\alpha_{012}\colon\ X_{01} \otimes X_{12} \longrightarrow X_{02}$ is the 1-simplex X_{jk} where $0 \leq j < k \leq 2$ are different to i. On the other hand the degenerated 1-simplex $s_0(\star)$ is equal to the object $\mathbb{1}$, the degenerated 2-simplices $s_0(X)$ and $s_1(X)$ are the morphisms l^{-1}_X and r^{-1}_X respectively, and the degenerated 3-simplices $s_0(\alpha_{012})$, $s_1(\alpha_{012})$

and $s_2(\alpha_{012})$ are the commutative diagrams:

$$\begin{tikzcd}
X_{02} & & \mathbb{1}\otimes X_{02} \arrow[ll, "l^{-1}_{X_{02}}"'] \\
 & & \mathbb{1}\otimes\big(X_{01}\otimes X_{12}\big) \arrow[u, "\mathbb{1}\otimes\alpha_{012}"'] \\
X_{01}\otimes X_{12} \arrow[uu, "\alpha_{012}"] & & \big(\mathbb{1}\otimes X_{01}\big)\otimes X_{12} \arrow[ll, "l^{-1}_{X_{01}}\otimes X_{12}"] \arrow[u, "a"', "\wr"]
\end{tikzcd}
\qquad
\begin{tikzcd}
X_{02} & & X_{01}\otimes X_{12} \arrow[ll, "\alpha_{012}"'] \\
 & & X_{01}\otimes\big(\mathbb{1}\otimes X_{12}\big) \arrow[u, "X_{01}\otimes l^{-1}_{X_{12}}"'] \\
X_{01}\otimes X_{12} \arrow[uu, "\alpha_{012}"] & & \big(X_{01}\otimes\mathbb{1}\big)\otimes X_{12} \arrow[ll, "r^{-1}_{X_{01}}\otimes X_{12}"] \arrow[u, "a"', "\wr"]
\end{tikzcd}$$

and

$$\begin{tikzcd}
X_{02} & & X_{01}\otimes X_{12} \arrow[ll, "\alpha_{012}"'] \\
 & & X_{01}\otimes\big(X_{12}\otimes\mathbb{1}\big) \arrow[u, "X_{01}\otimes r^{-1}_{X_{12}}"'] \\
X_{02}\otimes\mathbb{1} \arrow[uu, "r^{-1}_{X_{02}}"] & & \big(X_{01}\otimes X_{12}\big)\otimes\mathbb{1} \arrow[ll, "\alpha_{012}\otimes\mathbb{1}"] \arrow[u, "a"', "\wr"]
\end{tikzcd}$$

respectively.

From the next statement we deduce that the functor (28) is equal to the nerve functor for bicategories defined in [Duskin, 2002] when restricted to monoidal categories (*i.e.* bicategories with just one object).

3.6. Lemma. *The reduced simplicial set $\mathcal{N}\big(\mathcal{M}\big)$ is weakly 2-coskeletal for every monoidal category $\mathcal{M}$; in other words the function:*

$$\mathrm{Hom}_{\mathbf{sSet}}\big(\Delta^q,\mathcal{N}(\mathcal{M})\big)\longrightarrow\mathrm{Hom}_{\mathbf{sSet}}\big(\partial\Delta^q,\mathcal{N}(\mathcal{M})\big) \tag{30}$$

induced by the map $\partial\Delta^q\hookrightarrow\Delta^q$ is a bijection for $q\geq 4$ and an injection for $q=3$.

Proof. For $q=3$ the function (30) forgets the commutativity of the diagram (29), so it is an injection. We should show that for $q\geq 4$ the function:

$$\prod_s\delta^*_s\colon\ \mathcal{M}_q\longrightarrow\prod_{0\leq s\leq q}\mathcal{M}_{q-1}$$

is the kernel in the category **Set** of the arrows:

$$\prod_{0\leq s\leq q}\mathcal{M}_{q-1}\ \substack{\xrightarrow{\ \prod_{t<t'}\delta^*_t\circ\mathrm{proj}_{t'}\ }\\ \xrightarrow[\ \prod_{t<t'}\delta^*_{t'-1}\circ\mathrm{proj}_t\ ]{}}\ \prod_{0\leq t<t'\leq q}\mathcal{M}_{q-2}\ .$$

Explicitly giving a set of objects $\Big\{X^s_{ij}\ \Big|\ 0\leq i<j\leq q-1\,,\ 0\leq s\leq q\Big\}$ of $\mathcal{M}$ and a set of morphisms:

$$\Big\{\alpha^s_{ijk}\colon\ X^s_{ij}\otimes X^s_{jk}\longrightarrow X^s_{ik}\ \Big|\ 0\leq i<j<k\leq q-1\,,\ 0\leq s\leq q\Big\},$$

satisfying the following properties:

1. For $0 \leq i < j < k < l \leq q-1$ and $0 \leq s \leq q$ we have a commutative diagram:

$$\begin{tikzcd}
X^s_{il} & & X^s_{ij} \otimes X^s_{jl} \arrow[ll, "\alpha^s_{ijl}"'] \\
& & X^s_{ij} \otimes \left(X^s_{jk} \otimes X^s_{kl}\right) \arrow[u, "X^s_{ij}\otimes\alpha^s_{jkl}"'] \\
X^s_{ik} \otimes X^s_{kl} \arrow[uu, "\alpha^s_{ikl}"] & & \left(X^s_{ij} \otimes X^s_{jk}\right) \otimes X^s_{kl}\,, \arrow[ll, "\alpha^s_{ijk}\otimes X^s_{kl}"] \arrow[u, equal, "a\wr"]
\end{tikzcd} \tag{31}$$

2. $X^{t'}_{\delta_t a\,\delta_t b} \;=\; X^{t}_{\delta_{t'-1}a\,\delta_{t'-1}b}$ for every $0 \leq a < b \leq q-2$ and $0 \leq t < t' \leq q$,

3. $\alpha^{t'}_{\delta_t a\,\delta_t b\,\delta_t c} \;=\; \alpha^{t}_{\delta_{t'-1}a\,\delta_{t'-1}b\,\delta_{t'-1}c}$ for every $0 \leq a < b < c \leq q-2$ and $0 \leq t < t' \leq q$;

we shall prove that there exist a unique set $\left\{Y_{xy} \,\middle|\, 0 \leq x < y \leq q\right\}$ of objects of $\mathcal{M}$ and a unique set of morphisms:

$$\left\{\beta_{xyz}\colon\; Y_{xy} \otimes Y_{yz} \longrightarrow Y_{xz} \;\middle|\; 0 \leq x < y < z \leq q\right\}$$

which satisfy the following properties:

(iv) For $0 \leq i < j < k < l \leq q+1$ we have a commutative diagram:

$$\begin{tikzcd}
Y_{il} & & Y_{ij} \otimes Y_{jl} \arrow[ll, "\beta_{ijl}"'] \\
& & Y_{ij} \otimes \left(Y_{jk} \otimes Y_{kl}\right) \arrow[u, "Y_{ij}\otimes\beta_{jkl}"'] \\
Y_{ik} \otimes Y_{kl} \arrow[uu, "\beta_{ikl}"] & & \left(Y_{ij} \otimes Y_{jk}\right) \otimes Y_{kl}\,. \arrow[ll, "\beta_{ijk}\otimes Y_{kl}"] \arrow[u, equal, "a\wr"]
\end{tikzcd}$$

(v) $X^s_{ij} = Y_{\delta_s i\,\delta_s j}$ for every $0 \leq i < j \leq q$ and $0 \leq s \leq q+1$.

(vi) $\alpha^s_{ijk} = \beta_{\delta_s i\,\delta_s j\,\delta_s k}$ for every $0 \leq i < j < k \leq q$ and $0 \leq s \leq q+1$.

Notice that since $q \geq 4$ the properties (i), (v) and (vi) imply the property (iv) as well as the uniqueness of the objects Y_{xy} and the morphisms β_{xyz}. Hence it is enough to prove that the properties (ii) and (iii) imply the properties:

(vii) $X^s_{ij} = X^{s'}_{i'j'}$ for every $0 \leq i < j \leq q-1$, $0 \leq i' < j' \leq q-1$ and $0 \leq s, s' \leq q$ such that $\delta_s i = \delta_{s'} i'$ and $\delta_s j = \delta_{s'} j'$.

(viii) $\alpha^s_{ijk} = \alpha^{s'}_{i'j'k'}$ for every $0 \leq i < j < k \leq q-1$, $0 \leq i' < j' < k' \leq q-1$ and $0 \leq s, s' \leq q$ such that $\delta_s i = \delta_{s'} i'$, $\delta_s j = \delta_{s'} j'$ and $\delta_s k = \delta_{s'} k'$.

This is a consequence of the cases $n = 0, 1$ and $q \geq 4$ of the next easy to prove statement: Let $q \geq 2$ and $0 \leq n < q-1$. Given natural numbers $0 \leq i_0 < \cdots < i_n \leq q-1$, $0 \leq i'_0 < \cdots < i'_n \leq q-1$ and $0 \leq s < s' \leq q$ verifying that $\delta_s\, i_k = \delta_{s'}\, i'_k$ for every $0 \leq k \leq n$, there exist numbers $0 \leq a_0 < \cdots < a_n \leq q-2$ such that $\delta_{s'-1}\, a_k = i_k$ and $\delta_s\, a_k = i'_k$ for every $0 \leq k \leq n$. ∎

After some tedious calculations it follows from the Lemma 3.6 that the functor (28) is full and faithful. We need the also well known property:

3.7. Corollary. *For every* 2*-group* $\mathcal{G}$ *the reduced simplicial set* $\mathcal{N}(\mathcal{G})$ *is a Kan* 2*-group*[1]*; in other words the function:*

$$\mathrm{Hom}_{\mathbf{sSet}}\big(\Delta^q, \mathcal{N}(\mathcal{G})\big) \longrightarrow \mathrm{Hom}_{\mathbf{sSet}}\big(\Lambda^{q,k}, \mathcal{N}(\mathcal{G})\big) \tag{32}$$

induced by the map $\Lambda^{q,k} \hookrightarrow \Delta^q$ *is a surjection for* $q = 2$ *and* $0 \leq k \leq 2$ *and a bijection for* $q \geq 3$ *and* $0 \leq k \leq q$*. In particular* $\mathcal{N}(\mathcal{G})$ *is a reduced Kan complex such that* $\pi_i\big(\mathcal{N}(\mathcal{G})\big) = 0$ *for* $i \geq 3$*.*

Additionally a morphism of 2*-groups* $F\colon\ \mathcal{G} \longrightarrow \mathcal{G}'$ *is a weak equivalence if and only if the morphism of reduced simplicial sets* $\mathcal{N}(F)\colon\ \mathcal{N}\big(\mathcal{G}\big) \longrightarrow \mathcal{N}\big(\mathcal{G}'\big)$ *is a weak homotopy* ∞*-equivalence, if and only if* $\mathcal{N}(F)$ *is a weak homotopy* 2*-equivalence.*

Proof. We can deduce form the Lemma 3.6 that the function (32) is bijective for $q \geq 5$ and $0 \leq k \leq q$ (see for example the Lemma 1.7.1 of [Glenn, 1982]). It is easy to show directly that (32) is a surjective function for $q = 2$ and $0 \leq k \leq 2$ and that it is a bijective function for $q = 3$ and $0 \leq k \leq 3$. The function (32) is a bijection for $q = 4$ and $0 \leq k \leq 4$ because it forgets the conmutativity of one face of a cube (see also [Duskin, 2002]).

For the second part note that we can easily construct natural isomorphisms:

$$\begin{tikzcd} 2\text{-}\mathbf{Grp} \arrow[r, bend left, "\pi_{i+1}(\mathcal{N}(\,\cdot\,))"] \arrow[r, bend right, "\pi_i(\,\cdot\,)"'] \arrow[r, phantom, "\Downarrow"] & \mathbf{Grp} \end{tikzcd} \qquad \text{for} \quad 0 \leq i \leq 1\,,$$

using that $\mathcal{N}(\mathcal{G})$ is a Kan complex for every 2-group $\mathcal{G}$. ∎

It follows from the Main Theorem of [Duskin, 2002] that a simplicial set X is a Kan 2-group if and only if it is isomorphic to the geometric nerve of some 2-group.

3.8. The *geometrical* 2*-nerve functor for monoidal categories* (or *geometrical bi-nerve functor*) is the functor induced from the functors (26) and (27):

$$\mathbf{cat}^{\otimes}_{Nlax} \xrightarrow{\ \mathcal{N}^2\ } \mathbf{ssSet}_0 \qquad \text{where} \qquad \mathcal{N}^2\big(\mathcal{M}\big)_{p,q} \;=\; \big(\mathcal{M}^{[p]}\big)_q\,. \tag{33}$$

Explicitly if $\mathcal{M}$ is a monoidal category and $p, q \geq 0$ are natural numbers, then the set $\big(\mathcal{M}^{[p]}\big)_q$ of (p,q)-simplices of the geometrical 2-nerve of $\mathcal{M}$ is equal to the set of pairs $(X^\bullet, \alpha^\bullet)$ where:

$$X^\bullet = \Big\{\, X^\bullet_{ij}\colon\ [p] \longrightarrow \mathcal{M} \;\Big|\; 0 \leq i < j \leq q \,\Big\} \qquad \text{and}$$

---

[1] For $0 \leq n \leq \infty$ we call a simplicial set X a *Kan* n*-groupoid* or a n*-hypergroupoid* if the corresponding function (32) for X is a surjection for $2 \leq q \leq n$ and $0 \leq k \leq q$ and a bijection for $q \geq n+1$ and $0 \leq k \leq q$ (see [Duskin, 2002]). A *Kan* n*-group* is a reduced simplicial set that is a Kan n-groupoid.

$$\alpha^\bullet = \left\{ \alpha^\bullet_{ijk} \colon\; X^\bullet_{ij} \otimes X^\bullet_{jk} \Rightarrow X^\bullet_{ik} \;:\; [p] \to \mathcal{M} \;\middle|\; 0 \le i < j < k \le q \right\}$$

are families of functors and natural transformations respectively such that:

$$\begin{tikzcd}
X^\bullet_{il} & & X^\bullet_{ij} \otimes X^\bullet_{jl} \arrow[ll, Rightarrow, "\alpha^\bullet_{ijl}"'] \\
& & X^\bullet_{ij} \otimes \left(X^\bullet_{jk} \otimes X^\bullet_{kl}\right) \arrow[u, Rightarrow, "X^\bullet_{ij}\otimes\alpha^\bullet_{jkl}"'] \\
X^\bullet_{ik} \otimes X^\bullet_{kl} \arrow[uu, Rightarrow, "\alpha^\bullet_{ikl}"] & & \left(X^\bullet_{ij} \otimes X^\bullet_{jk}\right) \otimes X^\bullet_{kl} \arrow[ll, Rightarrow, "\alpha^\bullet_{ijk}\otimes X^\bullet_{kl}"] \arrow[u, equal, "a\wr"]
\end{tikzcd}$$

is a commutative diagram for $0 \le i < j < k < l \le q$

If we define $\underline{\mathcal{M}}_q$ as the category whose objects are the q-simplices (X, α) of $\mathcal{M}$ and where a morphism $f \colon\; (X,\alpha) \to (Y,\beta)$ is a family of morphisms of $\mathcal{M}$:

$$f = \left\{ f_{ij} \colon\; X_{ij} \longrightarrow Y_{ij} \;\middle|\; 0 \le i < j \le q \right\}$$

such that the following diagram is commutative:

$$\begin{tikzcd}
X_{ij} \otimes X_{jk} \arrow[r, "f_{ij}\otimes f_{jk}"] \arrow[d, "\alpha_{ijk}"'] & Y_{ij} \otimes Y_{jk} \arrow[d, "\beta_{ijk}"] \\
X_{ik} \arrow[r, "f_{ik}"'] & Y_{ik}
\end{tikzcd} \tag{34}$$

for $0 \le i < j < k \le q$; then it can be easily proved that there is a bijection of sets:

$$\mathcal{N}^2\big(\mathcal{M}\big)_{p,q} \;=\; \big(\mathcal{M}^{[p]}\big)_q \;\cong\; \mathrm{Hom}_{\mathbf{cat}}\big([p], \underline{\mathcal{M}}_q\big) \tag{35}$$

which is natural in the object $[p]$ of Δ^{op}.

We are ready to prove:

3.9. Theorem. *For every 2-group $\mathcal{G}$ the reduced bisimplicial set $\mathcal{N}^2\big(\mathcal{G}\big)$ is a fibrant object of the model category $\big(\mathbf{ssSet}_0, d^\star\mathbf{W}_2\big)$ of the Proposition 2.9.*

Proof. We shall prove that the reduced bisimplicial set $\mathcal{N}^2\big(\mathcal{G}\big)$ satisfies the properties 1-5 of the Corollary 2.10 for $n = 2$. First we note that 1 and 2 follow from Lemma 3.4 and Corollary 3.7.

On the other hand 3 is a consequence of the bijection (35) because we know that for $p \ge 3$ the function of totally ordered sets:

$$\bigsqcup_{0\le i\le p} \delta_i \colon\; \bigsqcup_{0\le i\le p} [p-1] \longrightarrow [p]$$

is the cokernel in **cat** of the arrows:

$$\bigsqcup_{0\le j<k\le p} [p-2] \underset{\underset{0\le j<k\le p}{\bigsqcup}(\mathrm{inc}_j\circ\delta_{k-1})}{\overset{\underset{0\le j<k\le p}{\bigsqcup}(\mathrm{inc}_k\circ\delta_j)}{\rightrightarrows}} \bigsqcup_{0\le i\le p} [p-1]\,.$$

To show that the property 4 of the Corollary 2.10 is satisfied we must prove that for $0 \leq k \leq 2$ the function:

$$\mathrm{Hom}_{\mathbf{ssSet}}\Big(\partial\Delta^2 \boxtimes \Delta^2, \mathcal{N}^2(\mathcal{G})\Big) \longrightarrow \mathrm{Hom}_{\mathbf{ssSet}}\Big(\partial\Delta^2 \boxtimes \Lambda^{2,k}, \mathcal{N}^2(\mathcal{G})\Big) \tag{36}$$

induced from the inclusion $\Lambda^{2,k} \hookrightarrow \Delta^2$ is surjective.

Notice that the domain of (36) can be identified with the set of triplets $(X^\bullet, \alpha^\bullet, f^\bullet)$ where:

$$X^\bullet = \left\{\, X^s_{ij} \;\middle|\; 0 \leq s \leq 2\,,\;\; 0 \leq i < j \leq 2 \,\right\}, \quad \alpha^\bullet = \left\{\, \alpha^s \colon\; X^s_{01} \otimes X^s_{12} \to X^s_{02} \;\middle|\; 0 \leq s \leq 2 \,\right\}$$
$$\text{and} \qquad f^\bullet = \left\{\, f^{st}_{ij} \colon\; X^s_{ij} \to X^t_{ij} \;\middle|\; 0 \leq s < t \leq 2\,,\;\; 0 \leq i < j \leq 2 \,\right\} \tag{37}$$

are families of objects and morphisms in $\mathcal{M}$ respectively:

$$\begin{tikzcd}
 & X^0_{01} \otimes X^0_{12} \arrow[rr, "\alpha^0"] \arrow[dl, "f^{01}_{01} \otimes f^{01}_{12}"'] \arrow[dd, "f^{02}_{01} \otimes f^{02}_{12}" near end] & & X^0_{02} \arrow[dl, "f^{01}_{02}"'] \arrow[dd, "f^{02}_{02}"] \\
X^1_{01} \otimes X^1_{12} \arrow[rr, "\alpha^1" near end] \arrow[dr, "f^{12}_{01} \otimes f^{12}_{12}"'] & & X^1_{02} \arrow[dr, "f^{12}_{02}"'] & \\
 & X^2_{01} \otimes X^2_{12} \arrow[rr, "\alpha^2"'] & & X^2_{02}\,,
\end{tikzcd} \tag{38}$$

verifying that $\alpha^t \circ \big(f^{st}_{01} \otimes f^{st}_{12}\big) = f^{st}_{02} \circ \alpha^s$ for every $0 \leq s < t \leq 2$. In the same way the codomain of (36) is identified with the set of pairs $(X^\bullet, f^\bullet)$ where:

$$\begin{gathered} X^\bullet = \left\{\, X^s_{ij} \;\middle|\; 0 \leq s \leq 2 \;\text{ and }\; (\, k = i < j \leq 2 \;\text{ or }\; 0 \leq i < j = k \,)\right\} \qquad \text{and} \\ f^\bullet = \left\{\, f^{st}_{ij} \colon\; X^s_{ij} \to X^t_{ij} \;\middle|\; 0 \leq s < t \leq 2 \;\text{ and }\; (\, k = i < j \leq 2 \;\text{ or }\; 0 \leq i < j = k \,)\,\right\} \end{gathered} \tag{39}$$

are families of objects and morphisms of $\mathcal{M}$ respectively. The function (36) forgets the following objects and morphisms:

$$\left\{\, X^s_{ij} \;\middle|\; 0 \leq s \leq 2\,,\;\; 0 \leq i < j \leq 2 \;\text{ and }\; i, j \neq k \,\right\},$$
$$\left\{\, f^{st}_{ij} \;\middle|\; 0 \leq s < t \leq 2 \;\text{ and }\; i, j \neq k \right\} \quad \text{and} \quad \left\{\, \alpha^s \;\middle|\; 0 \leq s \leq 2 \right\},$$

as well as the relations $\alpha^t \circ \big(f^{st}_{01} \otimes f^{st}_{12}\big) = f^{st}_{02} \circ \alpha^s$ for every $0 \leq s < t \leq 2$.

From this description it is easy to show that the function (36) is surjective for $k = 1$. It is surjective for $k = 0, 2$ by the Lemma 3.2.

To prove the property 5 we shall show that for $1 \leq n \leq 2$ and $0 \leq k \leq 2$ the function:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{ssSet}}\Big(\Delta^n \boxtimes \Delta^2, \mathcal{N}^2(\mathcal{G})\Big) \arrow[d] \\
\mathrm{Hom}_{\mathbf{ssSet}}\Big(\partial\Delta^n \boxtimes \Delta^2, \mathcal{N}^2(\mathcal{G})\Big) \underset{\mathrm{Hom}_{\mathbf{ssSet}}(\partial\Delta^n \boxtimes \Lambda^{2,k}, \mathcal{N}^2(\mathcal{G}))}{\times} \mathrm{Hom}_{\mathbf{ssSet}}\Big(\Delta^n \boxtimes \Lambda^{2,k}, \mathcal{N}^2(\mathcal{G})\Big)
\end{tikzcd} \tag{40}$$

induced from the commutative square:

$$\begin{tikzcd}
\Delta^n \times \Delta^2 & \Delta^n \times \Lambda^{2,k} \arrow[l, hook'] \\
\partial\Delta^n \times \Delta^2 \arrow[u, hook] & \partial\Delta^n \times \Lambda^{2,k} \arrow[l, hook'] \arrow[u, hook]
\end{tikzcd}$$

is surjective.

Case $n = 2$*:* The domain of the function (40) is identified with the set of triplets $(X^\bullet, \alpha^\bullet, f^\bullet)$ as in (37) such that $\alpha^t \circ \left(f^{st}_{01} \otimes f^{st}_{12}\right) = f^{st}_{02} \circ \alpha^s$ for every $0 \leq s < t \leq 2$ and $f^{12}_{ij} \circ f^{01}_{ij} = f^{02}_{ij}$ for every $0 \leq i < j \leq 2$. In the same way the codomain of (40) is identified with the set of triplets $(X^\bullet, \alpha^\bullet, f^\bullet)$ as in (37) satisfying that $\alpha^t \circ \left(f^{st}_{01} \otimes f^{st}_{12}\right) = f^{st}_{02} \circ \alpha^s$ for every $0 \leq s < t \leq 2$ and $f^{12}_{ij} \circ f^{01}_{ij} = f^{02}_{ij}$ for every $0 \leq i < j \leq 2$ where $i = k$ or $j = k$. The function (40) forgets the relation $f^{12}_{ij} \circ f^{01}_{ij} = f^{02}_{ij}$ where $i \neq k \neq j$.

It follows that (40) is bijection for $k = 1$ because $\mathcal{G}$ is a groupoid. We deduce that it is bijective for $k = 0, 2$ from the following easy to prove property: If $f\colon\ X \longrightarrow Y$ and $\xi\colon\ X \otimes X' \longrightarrow Y \otimes Y'$ are morphisms of a 2-group $\mathcal{G}$, there exists a unique morphism $f'\colon\ X' \longrightarrow Y'$ such that $f \otimes f' = \xi$.

Case $n = 1$*:* The domain of (40) is identified with the set of commutative squares of the type:

$$\begin{tikzcd}
X_2 \otimes X_0 \arrow[r, "f"] \arrow[d, "\varphi_2 \otimes \varphi_0"'] & X_1 \arrow[d, "\varphi_1"] \\
Y_2 \otimes Y_0 \arrow[r, "g"'] & Y_1\, ;
\end{tikzcd} \tag{41}$$

and the codomain with the set whose elements are the given by six morphism of $\mathcal{G}$ as follows:

$$X_2 \otimes X_0 \xrightarrow{f} X_1\ , \qquad Y_2 \otimes Y_0 \xrightarrow{g} Y_1\ , \qquad X_i \xrightarrow{\varphi_i} Y_i \quad \text{and} \quad X_j \xrightarrow{\varphi_j} Y_j$$

where $0 \leq i < j \leq 2$ and $i, j \neq k$. The function (40) forgets the morphism φ_k; then (40) is actually a bijection for $0 \leq k \leq 2$. ■

We want to show that the geometrical 2-nerve functor (33) is full and faithful. For that propose let is begin by proving that for every monoidal category $\mathcal{M}$ the bisimplicial set $\mathcal{N}^2(\mathcal{M})$ has an analogous property to that established in the Lemma 3.6 for the simplicial set $\mathcal{N}(\mathcal{M})$:

3.10. Let $\Delta \underset{3}{\times} \Delta$ be the full subcategory of the product category $\Delta \times \Delta$ whose objects are the pairs of totally ordered sets $\big([k], [l]\big)$ such that $0 \leq k + l \leq 3$. If we denote by $\mathbf{ssSet}_{\leq 3}$ the category of presheaves of sets over $\Delta \underset{3}{\times} \Delta$, then we obtain from the inclusion functor $\tau_3\colon\ \Delta \underset{3}{\times} \Delta \hookrightarrow \Delta \times \Delta$ an adjunction:

$$\begin{tikzcd}
\mathbf{ssSet}_{\leq 3} \arrow[r, bend right, "\tau_{3*}"'] \arrow[r, phantom, "\perp"] & \mathbf{ssSet} \arrow[l, bend right, "\tau_3^*"']
\end{tikzcd} \tag{42}$$

where the functor τ_3^* is just the truncation functor and τ_{3*} is a Kan extension which in this case is full and faithful.

Given a unit η of the adjunction (42) and a bisimplicial set X, it can be proved that the map of bisimplicial sets $\eta_X\colon\ X \longrightarrow \tau_{3*}\circ\tau_3^*(X)$ is an isomorphism if and only if the square:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{ssSet}}\big(\Delta^p\boxtimes\Delta^q, X\big) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathbf{ssSet}}\big(\partial\Delta^p\boxtimes\Delta^q, X\big) \arrow[d] \\
\mathrm{Hom}_{\mathbf{ssSet}}\big(\Delta^p\boxtimes\partial\Delta^q, X\big) \arrow[r] & \mathrm{Hom}_{\mathbf{ssSet}}\big(\partial\Delta^p\boxtimes\partial\Delta^q, X\big)
\end{tikzcd} \tag{43}$$

induced from the diagram of bisimplicial sets:

$$\begin{tikzcd}
\Delta^p\boxtimes\Delta^q & \partial\Delta^p\boxtimes\Delta^q \arrow[l, hook'] \\
\Delta^p\boxtimes\partial\Delta^q \arrow[u, hook] & \partial\Delta^p\boxtimes\partial\Delta^q \arrow[l, hook'] \arrow[u, hook]
\end{tikzcd} \tag{44}$$

is a cartesian square for every $p, q \geq 0$ such that $p+q>3$.

Let is prove that the bisimplicial set $\mathcal{N}^2(\mathcal{M})$ has this property for every monoidal category $\mathcal{M}$. Indeed it follows from Lemma 3.6 and the bijections (35) that if $X=\mathcal{N}^2(\mathcal{M})$ then the square (43) is cartesian for the pairs (p,q) such that $p\geq 3$ or $q\geq 4$. We should show that (43) is a cartesian square for the pairs $(2,2)$, $(1,3)$ and $(2,3)$.

Case $(p,q)=(2,2)$*:* We have that $\mathrm{Hom}_{\mathbf{ssSet}}\big(\partial\Delta^2\boxtimes\partial\Delta^2, \mathcal{N}^2(\mathcal{M})\big)$ can be identified with the set of pairs $\big(X^\bullet, f^\bullet\big)$ where:

$$X^\bullet=\big\{\,X_{ij}^s \,\big|\, 0\leq s\leq 2\,,\ \ 0\leq i<j\leq 2\,\big\} \qquad \text{and} \tag{45}$$

$$f^\bullet=\big\{\,f_{ij}^{st}\colon\ X_{ij}^s \to X_{ij}^t\ \,\big|\, 0\leq s<t\leq 2\,,\ \ 0\leq i<j\leq 2\,\big\}\,.$$

are families of objects and morphisms of $\mathcal{M}$ respectively. $\mathrm{Hom}_{\mathbf{ssSet}}\big(\Delta^2\boxtimes\partial\Delta^2, \mathcal{N}^2(\mathcal{M})\big)$ with the subset of $\mathrm{Hom}_{\mathbf{ssSet}}\big(\partial\Delta^2\boxtimes\partial\Delta^2, \mathcal{N}^2(\mathcal{M})\big)$ whose elements satisfy that $f_{ij}^{12}\circ f_{ij}^{01}=f_{ij}^{02}$ for every $0\leq i<j\leq 2$. As in the poof of the Theorem 3.9 $\mathrm{Hom}_{\mathbf{ssSet}}\big(\partial\Delta^2\boxtimes\Delta^2, \mathcal{N}^2(\mathcal{M})\big)$ can be identified with the set of triplets $\big(X^\bullet,\alpha^\bullet,f^\bullet\big)$ as in (37) such that $\alpha^t\circ\big(f_{01}^{st}\otimes f_{12}^{st}\big)=f_{02}^{st}\circ\alpha^s$ for every $0\leq s<t\leq 2$, and $\mathrm{Hom}_{\mathbf{ssSet}}\big(\Delta^2\boxtimes\Delta^2, \mathcal{N}^2(\mathcal{M})\big)$ with the subset of $\mathrm{Hom}_{\mathbf{ssSet}}\big(\partial\Delta^2\boxtimes\Delta^2, \mathcal{N}^2(\mathcal{M})\big)$ whose elements satisfy additionally that $f_{ij}^{12}\circ f_{ij}^{01}=f_{ij}^{02}$ for every $0\leq i<j\leq 2$.

In this case the horizontal functions of the square (43) are just inclusions and the verticals ones forget the family $\alpha^\bullet$ and the relations $\alpha^t\circ\big(f_{01}^{st}\otimes f_{12}^{st}\big)=f_{02}^{st}\circ\alpha^s$ for every $0\leq s<t\leq 2$. Then (43) is a cartesian square for $(p,q)=(2,2)$.

Case $(p,q)=(1,3)$*:* The set of morphisms $\mathrm{Hom}_{\mathbf{ssSet}}\big(\partial\Delta^1\boxtimes\partial\Delta^3, \mathcal{N}^2(\mathcal{M})\big)$ can be identify with the set of pairs $\big(X^\bullet,\alpha^\bullet\big)$ where:

$$X^\bullet=\big\{\,X_{ij}^s \,\big|\, 0\leq s\leq 1\,,\ 0\leq i<j\leq 3\,\big\} \quad \text{and} \tag{46}$$

$$\alpha^\bullet=\big\{\,\alpha_{ijk}^s\colon\ X_{ij}^s\otimes X_{jk}^s \to X_{ik}^s\ \,\big|\, 0\leq s\leq 1\,,\ 0\leq i<j<k\leq 3\,\big\}$$

are families of objects and morphisms of $\mathcal{M}$ respectively. $\mathrm{Hom}_{\mathbf{ssSet}}\big(\partial\Delta^1 \boxtimes \Delta^3, \mathcal{N}^2(\mathcal{M})\big)$ with the subset of $\mathrm{Hom}_{\mathbf{ssSet}}\big(\partial\Delta^1 \boxtimes \partial\Delta^3, \mathcal{N}^2(\mathcal{M})\big)$ of such pairs $\big(X^\bullet, \alpha^\bullet\big)$ satisfying that for every $0 \leq s \leq 1$ the diagram (31) is commutative where $i = 0$, $j = 1$, $k = 2$ and $l = 3$. $\mathrm{Hom}_{\mathbf{ssSet}}\big(\Delta^1 \boxtimes \partial\Delta^3, \mathcal{N}^2(\mathcal{M})\big)$ can be identify with the set of triplets $\big(X^\bullet, \alpha^\bullet, f_\bullet\big)$ where $\big(X^\bullet, \alpha^\bullet\big)$ is a pair like in (46) and $f_\bullet = \big\{\, f_{ij} \colon\ X^0_{ij} \to X^1_{ij}\ \big|\ 0 \leq i < j \leq 3 \,\big\}$ is a family of morphisms of $\mathcal{M}$ such that $\alpha^1_{ijk} \circ \big(f_{ij} \otimes f_{jk}\big) = f_{ik} \circ \alpha^0_{ijk}$ for every $0 \leq i < j < k \leq 3$. And finally $\mathrm{Hom}_{\mathbf{ssSet}}\big(\Delta^1 \boxtimes \Delta^3, \mathcal{N}^2(\mathcal{M})\big)$ can be identify with the subset of $\mathrm{Hom}_{\mathbf{ssSet}}\big(\Delta^1 \boxtimes \partial\Delta^3, \mathcal{N}^2(\mathcal{M})\big)$ whose elements verify additionally that for every $0 \leq s \leq 1$ the diagram (31) is commutative where $i = 0$, $j = 1$, $k = 2$ and $l = 3$.

In the diagram (43) the verticals functions are just inclusions and the horizontals ones forgets the family of morphisms $f_\bullet$ together with the relations $\alpha^1_{ijk} \circ \big(f_{ij} \otimes f_{jk}\big) = f_{ik} \circ \alpha^0_{ijk}$ for every $0 \leq i < j < k \leq 3$. We conclude that (43) is a cartesian square for $(p, q) = (1, 3)$.

Case $(p, q) = (2, 3)$*:* In this case we see that $\mathrm{Hom}_{\mathbf{ssSet}}\big(\partial\Delta^2 \boxtimes \partial\Delta^3, \mathcal{N}^2(\mathcal{M})\big)$ can be identified with the set of triplets $\big(X^\bullet, \alpha^\bullet, f^\bullet\big)$ where:

$$X^\bullet = \big\{\, X^s_{ij} \,\big|\, 0 \leq s \leq 2\,,\ \ 0 \leq i < j \leq 3 \,\big\}\,, \tag{47}$$

$$\alpha^\bullet = \big\{\, \alpha^s_{ijk} \colon\ X^s_{ij} \otimes X^s_{jk} \to X^s_{ik}\ \big|\, 0 \leq s \leq 2\,,\ \ 0 \leq i < j < k \leq 3 \,\big\}$$

$$\text{and} \qquad f^\bullet = \big\{\, f^{st}_{ij} \colon\ X^s_{ij} \to X^t_{ij}\ \big|\, 0 \leq s < t \leq 2\,,\ \ 0 \leq i < j \leq 3 \,\big\}$$

are families of objects and morphisms of $\mathcal{M}$ respectively, such that $\alpha^t_{ijk} \circ \big(f^{st}_{ij} \otimes f^{st}_{jk}\big) = f^{st}_{ik} \circ \alpha^s_{ijk}$ for every $0 \leq s < t \leq 2$ and $0 \leq i < j < k \leq 3$. $\mathrm{Hom}_{\mathbf{ssSet}}\big(\partial\Delta^2 \boxtimes \Delta^3, \mathcal{N}^2(\mathcal{M})\big)$ with the subset of $\mathrm{Hom}_{\mathbf{ssSet}}\big(\partial\Delta^2 \boxtimes \partial\Delta^3, \mathcal{N}^2(\mathcal{M})\big)$ whose elements make the diagram (31) commute for every $0 \leq s \leq 2$ where $i = 0$, $j = 1$, $k = 2$ and $l = 3$, and $\mathrm{Hom}_{\mathbf{ssSet}}\big(\Delta^2 \boxtimes \partial\Delta^3, \mathcal{N}^2(\mathcal{M})\big)$ with the subset of $\mathrm{Hom}_{\mathbf{ssSet}}\big(\partial\Delta^2 \boxtimes \partial\Delta^3, \mathcal{N}^2(\mathcal{M})\big)$ whose elements verify that $f^{12}_{ij} \circ f^{01}_{ij} = f^{02}_{ij}$ for every $0 \leq i < j \leq 3$.

Such $\mathrm{Hom}_{\mathbf{ssSet}}\big(\Delta^2 \boxtimes \Delta^3, \mathcal{N}^2(\mathcal{M})\big)$ can be identified with the subset of $\mathrm{Hom}_{\mathbf{ssSet}}\big(\partial\Delta^2 \boxtimes \partial\Delta^3, \mathcal{N}^2(\mathcal{M})\big)$ whose elements verify the identity $f^{12}_{ij} \circ f^{01}_{ij} = f^{02}_{ij}$ for every $0 \leq i < j \leq 3$ and that the diagram (31) is commutative for every $0 \leq s \leq 2$ where $i = 0$, $j = 1$, $k = 2$ and $l = 3$, it follows that (43) is a cartesian square for $(p, q) = (2, 3)$.

Let is note finally that for the pairs $(0, 3)$, $(1, 2)$, $(2, 1)$ and $(3, 0)$ the function:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{ssSet}}\big(\Delta^n \boxtimes \Delta^n, \mathcal{N}^2(\mathcal{G})\big) \arrow[d] \\
\mathrm{Hom}_{\mathbf{ssSet}}\big(\partial\Delta^n \boxtimes \Delta^n, \mathcal{N}^2(\mathcal{G})\big) \underset{\mathrm{Hom}_{\mathbf{ssSet}}(\partial\Delta^n \boxtimes \partial\Delta^n, \mathcal{N}^2(\mathcal{G}))}{\times} \mathrm{Hom}_{\mathbf{ssSet}}\big(\Delta^n \boxtimes \partial\Delta^n, \mathcal{N}^2(\mathcal{G})\big)
\end{tikzcd} \tag{48}$$

induced from (44) is an injection. Indeed for the pairs $(3, 0)$ it is trivially a bijection and for the pairs $(0, 3)$, $(1, 2)$ and $(2, 1)$ the function (48) just forgets the commutativity of some diagrams, then is an injection.

We then have:

3.11. Lemma. *The geometrical 2-nerve functor* $\mathcal{N}^2\colon\ \mathbf{cat}^{\otimes}_{Nlax} \longrightarrow \mathbf{ssSet}_0$ *from* (33) *is full and faithful.*

Proof. Recall that $\mathcal{N}\colon\ \mathbf{cat}^{\otimes}_{Nlax} \longrightarrow \mathbf{sSet}_0$ the geometric nerve functor (28) is full and faithful. It follows that the functor $\mathcal{N}^2$ is faithful because $\mathcal{N}^2(\mathcal{M})_{0,q} = \mathcal{N}(\mathcal{M})_q$ for every monoidal category $\mathcal{M}$ and every $q \geq 0$.

Moreover given a morphism of bisimplicial sets $\varphi\colon\ \mathcal{N}^2(\mathcal{M}) \longrightarrow \mathcal{N}^2(\mathcal{M}')$ there existe a normal lax monoidal functor $F\colon\ \mathcal{M} \longrightarrow \mathcal{M}'$ such that $\mathcal{N}^2(F)_{0,q} = \varphi_{0,q}$ for every $q \geq 0$. It follows from the property proved before this Lemma that to show the equality $\mathcal{N}^2(F) = \varphi$ we just need to verify that $\mathcal{N}^2(F)_{1,1}(\alpha) = \varphi_{1,1}(\alpha)$ for every morphism $\alpha\colon\ X \longrightarrow Y$ of $\mathcal{M}$.

Firs notice that $\mathcal{N}^2(F)_{1,1}(s_0^h X) = s_0^h \circ \mathcal{N}^2(F)_{0,1}(X) = s_0^h \circ \varphi_{0,1}(X) = \varphi_{1,1}(s_0^h X)$ for every object X of $\mathcal{M}$. Let $\alpha\colon\ X \longrightarrow Y$ be an arbitrary morphism of $\mathcal{M}$ and consider the following commutative square:

$$\begin{tikzcd}
X \otimes \mathbb{1} \arrow[r, "r_X^{-1}"] \arrow[d, "X\otimes\mathbb{1}"'] & X \arrow[d, "\alpha"] \\
X \otimes \mathbb{1} \arrow[r, "\alpha\circ r_X^{-1}"'] & Y
\end{tikzcd}$$

seen as an element of $\mathcal{N}^2(\mathcal{M})_{1,2}$ and apply to it the function $\varphi_{1,2}$. It follows that:

$$\begin{tikzcd}
F(X)\otimes\mathbb{1} \arrow[r, "m^F_{X,\mathbb{1}}"] \arrow[rr, bend left, "r^{-1}_{F(X)}"] \arrow[d, "F(X)\otimes\mathbb{1}"'] & F(X\otimes\mathbb{1}) \arrow[r, "F(r_X^{-1})"] & F(X) \arrow[d, "\varphi_{1,1}(\alpha)"] \\
F(X)\otimes\mathbb{1} \arrow[r, "m^F_{X,\mathbb{1}}"] \arrow[rr, bend right, "F(\alpha)\circ r^{-1}_{F(X)}"'] & F(X\otimes\mathbb{1}) \arrow[r, "F(\alpha\circ r_X^{-1})"] & F(Y)
\end{tikzcd}$$

is a commutative diagram of $\mathcal{M}'$. Then $\varphi_{1,1}(\alpha) = F(\alpha) = \mathcal{N}^2(F)_{1,1}(\alpha)$. ■

3.12. Let is denote by $\mathbf{scat}_0$ the full subcategory of the category of functors $\mathbf{cat}^{\mathbf{\Delta^{op}}}$ where the objects are the functors $X_\bullet\colon\ \Delta^{op} \to \mathbf{cat}$ such that $X_0 = \star$ is the category with one object and one morphism. From the geometrical 2-nerve functor (33) and the bijection (35) we obtain a unique functor:

$$\mathbf{cat}^{\otimes}_{Nlax} \xrightarrow{\mathcal{N}^{cat}} \mathbf{scat}_0 \qquad \text{defined in objects as} \qquad \mathcal{N}^{cat}\big(\mathcal{M}\big)_q = \underline{\mathcal{M}}_q\,, \tag{49}$$

and a natural isomorphism $\mathcal{N}^2 \Rightarrow \mathrm{N}^{\Delta^{op}} \circ \mathcal{N}^{cat}$ defined by (35), where $\mathrm{N}\colon\ \mathbf{cat} \longrightarrow \mathbf{sSet}$ is the usual geometric nerve functor for small categories $\mathrm{N}(\mathcal{A})_p = \mathrm{Hom}_{\mathbf{cat}}([p], \mathcal{A})$.

We call (49) the *geometrical* **cat**-*nerve functor for monoidal categories.* Note that if $F\colon\ \mathcal{M} \longrightarrow \mathcal{M}'$ is a normal lax monoidal functor between monoidal categories and $q \geq 0$, the functor $\mathcal{N}^{cat}(F)_q$ is defined in a morphism of q-simplices $f = \{f_{ij}\}$ by $\mathcal{N}^{cat}(F)_q(f)_{ij} = F(f_{ij})$ for $0 \leq i < j \leq q$.

Recall that the structure of 2-category $\underline{\mathbf{cat}}$ of $\mathbf{cat}$ induces a structure of 2-category $\underline{\mathbf{scat}}_0$ in $\mathbf{scat}_0$ defined as follows: If $X_\bullet, Y_\bullet\colon\ \Delta^{op}\longrightarrow \mathbf{cat}$ are functors such that $X_0=\star=Y_0$ and $\alpha_\bullet,\beta_\bullet\colon\ X_\bullet\longrightarrow Y_\bullet$ are natural transformations, then a 2-arrow $\varepsilon\colon\ \alpha_\bullet\Rightarrow\beta_\bullet$ is a family of natural transformations $\left\{\,\varepsilon\colon\ \alpha_q\Longrightarrow\beta_q\,\right\}_{q\geq 0}$ such that $\varepsilon_q\star\varphi^*=\varphi^*\star\varepsilon_{q'}$ for every morphism $\varphi\colon\ [q']\longrightarrow[q]$ of Δ.

The *geometrical* **cat**-*nerve* 2-*functor for monoidal categories* is the 2-functor:

$$\underline{\mathbf{cat}}^{\otimes}_{Nlax}\xrightarrow{\underline{\mathcal{N}}^{cat}}\underline{\mathbf{scat}}_0 \tag{50}$$

whose underlying functor is (49) and where the image of a monoidal natural transformation of normal lax monoidal functors $\eta\colon\ F\Rightarrow G\ :\ \mathcal{M}\longrightarrow\mathcal{M}'$ is the 2-arrow $\mathcal{N}^{cat}(\eta)$ defined for every $q\geq 0$ and every q-simplex (X,α) of $\mathcal{M}$ as the morphism of q-simplices $\big(\mathcal{N}^{cat}(\eta)_q\big)_{(X,\alpha)}\colon\ F(X,\alpha)\longrightarrow G(X,\alpha)$ given by the family of morphisms of $\mathcal{M}'$:

$$\left\{\,\eta_{X_{ij}}\colon\ F(X_{ij})\longrightarrow G(X_{ij})\ \middle|\, 0\leq i<j\leq q\right\}.$$

3.13. Lemma. *The* 2-*functor* (50) *is full and faithful in the sense that the functor:*

$$\underline{\mathrm{Hom}}_{\mathbf{cat}^{\otimes}_{Nlax}}\big(\mathcal{M},\,\mathcal{M}'\big)\longrightarrow\underline{\mathrm{Hom}}_{\mathbf{scat}_0}\big(\mathcal{N}^{cat}(\mathcal{M}),\,\mathcal{N}^{cat}(\mathcal{M}')\big) \tag{51}$$

is an isomorphism of categories for every monoidal categories $\mathcal{M}$ *and* $\mathcal{M}'$.

Proof. By Lemma 3.11 the functor (51) is a bijection in objects, because (35) is a natural isomorphism $\mathcal{N}^2\ \Rightarrow\ \mathrm{N}^{\Delta^{op}}\circ\mathcal{N}^{cat}$ where N is a full and faithful functor

To prove that (51) is a full and faithful functor let is consider two fixed objects $\mathcal{N}^{cat}(F),\mathcal{N}^{cat}(G)\colon\ \mathcal{N}^{cat}(\mathcal{M})\longrightarrow\mathcal{N}^{cat}(\mathcal{M}')$ of the target category of (51). Recall that for every monoidal category $\mathcal{M}$, a unit of the adjunction (42) induces an isomorphism of bisimplicial sets $\mathcal{N}^2(\mathcal{M})\ \cong\ \tau_{3*}\tau_3^*\big(\mathcal{M}\big)$, and (35) induces an isomorphism $\mathcal{N}^2(\mathcal{M})\ \cong\ \mathrm{N}^{\Delta^{op}}\circ\mathcal{N}^{cat}(\mathcal{M})$. It follows that a 2-arrow $\epsilon\ :\ \mathcal{N}^{cat}(F)\Rightarrow\mathcal{N}^{cat}(G)$ is given by a natural transformation $\eta\colon\ =\epsilon_1\colon\ F\Rightarrow G$ and a function $\epsilon_2\colon\ \mathrm{Obj}\big(\underline{\mathcal{M}}_2\big)\longrightarrow\mathrm{Mor}\big(\underline{\mathcal{M}}'_2\big)$ such that:

1. $\eta_{\mathbb{1}_{\mathcal{M}}}=\mathrm{id}_{\mathbb{1}_{\mathcal{M}'}}$.

2. If $\alpha_{012}\colon\ X_{01}\otimes X_{12}\longrightarrow X_{02}$ is a 2-simplex of $\mathcal{M}$ the morphism $\epsilon_2(\alpha)$ of 2-simplices of $\mathcal{M}'$ is just the commutative diagram of $\mathcal{M}'$:

$$\begin{tikzcd}
F(X_{01})\otimes F(X_{12}) \arrow[r, "m^F_{X_{01},X_{12}}"] \arrow[d, "\eta_{X_{01}}\otimes\eta_{X_{12}}"'] & F(X_{01}\otimes X_{12}) \arrow[r, "F(\alpha_{012})"] & F(X_{02}) \arrow[d, "\eta_{X_{02}}"] \\
G(X_{01})\otimes G(X_{12}) \arrow[r, "m^G_{X_{01},X_{12}}"'] & G(X_{01}\otimes X_{12}) \arrow[r, "G(\alpha_{012})"'] & G(X_{02})
\end{tikzcd} \tag{52}$$

In other words a morphism $\mathcal{N}^{cat}(F) \Rightarrow \mathcal{N}^{cat}(G)$ in the target category of (51) is given by a natural transformation $\eta\colon F \Rightarrow G$ such that $\eta_{\mathbb{1}_{\mathcal{M}}} = \mathrm{id}_{\mathbb{1}_{\mathcal{M}'}}$ and for every morphism $\alpha_{012}\colon X_{01} \otimes X_{12} \to X_{02}$ of $\mathcal{M}$ the diagram (52) is commutative.

Taking the morphism α_{012} in the property 2 as the identity of the product $X \otimes Y$ for every objects X and Y of $\mathcal{M}$, it follows that η is a monoidal natural transformation between the normal lax monoidal functors $F, G\colon \mathcal{M} \longrightarrow \mathcal{M}'$. Conversely if $\eta\colon F \Rightarrow G$ is a monoidal natural transformation and $\alpha_{012}\colon X_{01} \otimes X_{12} \longrightarrow X_{02}$ is a morphism of $\mathcal{M}$, then the diagram (52) is commutative because we have the commutative squares of $\mathcal{M}'$:

$$\begin{tikzcd}
F(X_{01}) \otimes F(X_{12}) \arrow[r, "m^F_{X_{01},X_{12}}"] \arrow[d, "\eta_{X_{01}} \otimes \eta_{X_{12}}"'] & F(X_{01} \otimes X_{12}) \arrow[d, "\eta_{X_{01} \otimes X_{12}}"] \\
G(X_{01}) \otimes G(X_{12}) \arrow[r, "m^G_{X_{01},X_{12}}"'] & G(X_{01} \otimes X_{12}))
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
F(X_{01} \otimes X_{12}) \arrow[r, "F(\alpha_{012})"] \arrow[d, "\eta_{X_{01} \otimes X_{12}}"'] & F(X_{02}) \arrow[d, "\eta_{X_{02}}"] \\
G(X_{01} \otimes X_{12}) \arrow[r, "G(\alpha_{012})"'] & G(X_{02})\,.
\end{tikzcd}$$

Therefore (51) is a full and faithful functor. ■

Recall that the geometric nerve functor for small categories $\mathrm{N}\colon \mathbf{cat} \to \mathbf{sSet}$ has an extension to a simplicial functor:

$$\mathrm{N}\big(\mathcal{B}^{\mathcal{A}}\big) \longrightarrow \underline{\mathrm{Hom}}_{\mathbf{sSet}}\big(\mathrm{N}(\mathcal{A}), \mathrm{N}(\mathcal{B})\big) \tag{53}$$

where $\underline{\mathrm{Hom}}_{\mathbf{sSet}}\big(\mathrm{N}(\mathcal{A}), \mathrm{N}(\mathcal{B})\big)_k = \mathrm{Hom}_{\mathbf{sSet}}\big(\mathrm{N}(\mathcal{A}) \times \Delta^k, \mathrm{N}(\mathcal{B})\big)$ for every $k \geq 0$ and $\mathcal{B}^{\mathcal{A}}$ is the category of functors and naturals transformations. Moreover (53) is an isomorphism of simplicial sets for every categories $\mathcal{A}$ and $\mathcal{B}$.

From this fact and Lemma 3.13 we have:

3.14. Corollary. *The full and faithful geometrical 2-nerve functor for monoidal categories* $\mathcal{N}^2\colon \mathbf{cat}^{\otimes}_{Nlax} \longrightarrow \mathbf{ssSet}_0$ *from* (33) *has an extension to a simplicial funtor:*

$$\mathrm{N}\big(\underline{\mathrm{Hom}}_{\mathbf{cat}^{\otimes}_{Nlax}}(\mathcal{M}, \mathcal{M}')\big) \longrightarrow \underline{\mathrm{Hom}}^{(1)}_{\mathbf{ssSet}_0}\big(\mathcal{N}^2(\mathcal{M}), \mathcal{N}^2(\mathcal{M}')\big) \tag{54}$$

where $\underline{\mathrm{Hom}}^{(1)}_{\mathbf{ssSet}_0}(X, Y)_n = \mathrm{Hom}_{\mathbf{ssSet}}\big(X \times p_1^*(\Delta^n), Y\big)$ *and* (54) *is an isomorphism of simplicial sets.*

To conclude notice that the geometrical **cat**-nerve 2-functor for monoidal categories (50) is not the restriction to $\underline{\mathbf{cat}}^{\otimes}_{Nlax}$ of the nerve 2-functor for bicategories defined in [Lack, Paoli, 2008]. Nevertheless the 2-functor (50) and the nerve 2-functor of [Lack, Paoli, 2008] are equal when restricted to the 2-category 2-$\underline{\mathbf{Grp}}$ of 2-groups.

4. Deligne's determinant functors

4.1. Denote by **Grp** the full subcategory of **cat** whose objects are the small groups and by $\mathrm{B}\colon \mathbf{Grp} \longrightarrow \mathbf{sSet}_0$ the usual geometric nerve functor for small categories restricted to groups: $\mathrm{B}(G)_m = G^m$.

4.2. Lemma. *If G is a group and X is a reduced simplicial set, $\underline{\mathrm{Hom}}_{\mathbf{sSet}_0}\big(X, \mathrm{B}(G)\big)$ the simplicial set defined in (2), is a constant simplicial set:*

$$\pi_0\big(\underline{\mathrm{Hom}}_{\mathbf{sSet}_0}\big(X, \mathrm{B}(G)\big)\big) = \mathrm{Hom}_{\mathbf{sSet}_0}\big(X, \mathrm{B}(G)\big)\,.$$

Proof. Consider a morphism $F\colon\ X \times \Delta^n \longrightarrow \mathrm{B}(G)$ such that $F_m(\star, \varphi) = (e, \ldots, e) \in G^m$ for every $m \geq 1$ and $\varphi \in \Delta^n_m$. We will prove that $F_m(\xi, \varphi) = F_m(\xi, \psi)$ for every $m \geq 1$, $\xi \in X_m$ and $\varphi, \psi \in \Delta^n_m$. In fact as the reduced simplicial set $\mathrm{B}(G)$ is weakly 1-coskeletal (in the sense of the Lemma 3.6, see [Duskin, 2002]) we just need to prove the equality $F_1(\alpha, \varphi) = F_1(\alpha, \psi)$ for every $\alpha \in X_1$ and $\varphi, \psi \in \Delta^n_1$.

Notice that for every $\eta \in X_2$ and $\varphi \in \Delta^n_2$ the 2-simplex $F_2(\eta, \varphi)$ says that the element $F_1(d_1\eta, \varphi \circ \delta_1)$ of G is equal to the product $F_1(d_2\eta, \varphi \circ \delta_2) \cdot F_1(d_0\eta, \varphi \circ \delta_0)$. Then for every $\alpha \in X_1$ and every $\varphi \in \Delta^n_1$ it follows from the 2-simplices $F_2(s_0\alpha, \varphi \circ \sigma_1)$ and $F_2(s_1\alpha, \varphi \circ \sigma_0)$ that:

$$F_1(\alpha, \varphi \circ \delta_1 \circ \sigma_0) \;=\; F_1(\alpha, \varphi) \;=\; F_1(\alpha, \varphi \circ \delta_0 \circ \sigma_0)\,.$$

It is straightforward to show that $F_1(\alpha, \varphi) = F_1(\alpha, \psi)$ for every $\varphi, \psi \in \Delta^n_1$. ■

Let X be a reduced simplicial set and G be a group. An *additive function of X with values in G* is a function $\mathfrak{a}\colon\ X_1 \longrightarrow G$ with the properties:

1. (Unit) $\mathfrak{a}(s_0\star)$ is the identity element of G.
2. (Additivity) For every $\alpha \in X_2$ the product $\mathfrak{a}\big(d_2(\alpha)\big) \cdot \mathfrak{a}\big(d_0(\alpha)\big)$ is equal to $\mathfrak{a}\big(d_1(\alpha)\big)$.

If we denote by $\mathbf{add}_X(G)$ the set of all the additive functions of X with values in G, we have a functor:

$$\big(\mathbf{sSet}_0\big)^{op} \times \mathbf{Grp} \xrightarrow{\mathbf{add}_{\,\cdot}(\,\cdot\,)} \mathbf{Set} \tag{55}$$

which is defined in a morphism of reduced simplicial sets $f\colon\ Y \longrightarrow X$ and a homomorphism of groups $\varphi\colon\ G \longrightarrow H$ by the function $\mathbf{add}_f(\varphi)\colon\ \mathbf{det}_X(G) \longrightarrow \mathbf{det}_Y(H)$ where $\mathbf{add}_f(\varphi)(\mathfrak{a}) = \varphi \circ \mathfrak{a} \circ f_1$.

4.3. Lemma. *The functor (55) is naturally isomorphic to the functor* $\mathrm{Hom}_{\mathbf{sSet}_0}(\,\cdot\,,\ \mathrm{B}(\,\cdot\,))$.

Proof. Since the reduced simplicial set $\mathrm{B}(G)$ is weakly 1-coskeletal, it follows easily that the natural function $\mathrm{Hom}_{\mathbf{sSet}_0}\big(X, \mathrm{B}(G)\big) \longrightarrow \mathbf{add}_X(G)$ defined by $f_\bullet \mapsto f_1$ is a bijection. ■

From the Proposition 2.1 we known that $\big(\mathbf{sSet}_0, \nu^\star\mathbf{W}_1, \underline{\mathrm{Hom}}_{\mathbf{sSet}_0}\big)$ is a simplicial model category. Moreover $\mathrm{B}(G)$ is a fibrant object of the model category $(\mathbf{sSet}_0, \nu^\star\mathbf{W}_1)$ as a reduced Kan complex whose homotopy groups π_i are zero for $i \geq 2$. It follows from the Lemmas 4.2 and 4.3 that the set $\mathbf{add}_X(G)$ is naturally isomorphic to the set of maps from X to $\mathrm{B}(G)$ in the homotopy category $\mathbf{sSet}_0\big[(\nu^\star\mathbf{W}_1)^{-1}\big]$ of reduced homotopy 1-types (or even in the homotopy category $\mathbf{sSet}_\star\big[(\pi^\star\mathbf{W}_1)^{-1}\big]$ of pointed homotopy 1-types using the left adjoint of the adjunction (9)).

Note that the induced functor $\widetilde{\mathrm{B}}\colon\ \mathbf{Grp} \longrightarrow \mathbf{sSet}_0\big[(\nu^\star\mathbf{W}_1)^{-1}\big]$ from the geometric nerve functor for groups is an equivalence of categories whose inverse is induced from the fundamental group functor π_1 for reduced simplicial sets. Therefore we have:

4.4. Corollary. *For every reduced simplicial set X the functor $\mathbf{add}_X\colon\ \mathbf{Grp} \longrightarrow \mathbf{Set}$ is representable by the fundamental group $\pi_1(X)$ of X. In particular $f\colon\ Y \longrightarrow X$ is a weak homotopy 1-equivalence of reduced simplicial sets if and only if for every group G the function $\mathbf{add}_f(G)\colon\ \mathbf{add}_X(G) \longrightarrow \mathbf{add}_Y(G)$ is a bijection.*

4.5. Let $\mathcal{G}$ be a 2-group and X a reduced simplicial set. A *determinant of X with values in $\mathcal{G}$* is a pair of functions $D = (D, T)$:

$$X_1 \xrightarrow{\ D\ } \big\{\text{Objets of } \mathcal{G}\big\} \qquad \text{and} \qquad X_2 \xrightarrow{\ T\ } \big\{\text{Morphisms of } \mathcal{G}\big\}$$

with the following properties:

1. (Compatibility) For every $\xi \in X_2$ the morphism $T(\xi)$ of $\mathcal{G}$ is of the type:

$$D(d_2\xi) \otimes D(d_0\xi) \xrightarrow{T(\xi)} D(d_1\xi)\,.$$

2. (Unit) $D(s_0 \star) = \mathbb{1}$ and $T(s_0 \circ s_0 \star) = l_{\mathbb{1}}^{-1} = r_{\mathbb{1}}^{-1}$.
3. (Associativity) For every $\eta \in X_3$ we have the commutative diagram:

$$\begin{tikzcd}
D(A_{03}) & & D(A_{01}) \otimes D(A_{13}) \arrow[ll, "T(d_2\,\eta)"'] \\
 & & D(A_{01}) \otimes (D(A_{12}) \otimes D(A_{23})) \arrow[u, "D(A_{01})\otimes T(d_0\,\eta)"'] \\
D(A_{02}) \otimes D(A_{23}) \arrow[uu, "T(d_1\,\eta)"] & & (D(A_{01}) \otimes D(A_{12})) \otimes D(A_{23})\,, \arrow[ll, "T(d_3\,\eta)\otimes D(A_{23})"] \arrow[u, "a", "\wr"']
\end{tikzcd} \tag{56}$$

$$\text{where} \qquad \begin{array}{llll} A_{03} = d_1d_1\eta = d_1d_2\eta & & A_{01} = d_2d_2\eta = d_2d_3\eta \\ A_{13} = d_1d_0\eta = d_0d_2\eta & & A_{02} = d_2d_1\eta = d_1d_3\eta \\ A_{23} = d_0d_0\eta = d_0d_1\eta & & A_{12} = d_2d_0\eta = d_0d_3\eta\,. \end{array}$$

We denote by $\mathbf{det}_X(\mathcal{G})$ the set of the determinants of X with values in $\mathcal{G}$. Notice that there is a functor:

$$\big(\mathbf{sSet}_0\big)^{op} \times 2\text{-}\mathbf{Grp} \xrightarrow{\mathbf{det}_{\,\cdot}(\,\cdot\,)} \mathbf{Set} \tag{57}$$

which is defined in the morphisms $f\colon\ Y \longrightarrow X$ and $F = (F, m^F)\colon\ \mathcal{G} \longrightarrow \mathcal{H}$ by the function $\mathbf{det}_f(F)(D,T) = (\overline{D}, \overline{T})$ where $\overline{D}(B) = F\big(D(f_1\,B)\big)$ and $\overline{T}(\tau) = F\big(T(f_2\tau)\big) \circ m^F$.

Denote by $\mathcal{B}\colon\ 2\text{-}\mathbf{Grp} \longrightarrow \mathbf{sSet}_0$ the geometric nerve functor for monoidal categories (28) when it is restricted to the category of 2-groups.

4.6. Lemma. *The functor (57) is naturally isomorphic to the functor* $\mathrm{Hom}_{\mathbf{sSet}_0}\big(\,\cdot\,,\,\mathcal{B}(\,\cdot\,)\big)$.

Proof. For every reduced simplicial set X and every 2-group $\mathcal{G}$ we have a well defined function:

$$\begin{array}{ccc} \operatorname{Hom}_{\mathbf{sSet}_0}\big(X,\mathcal{B}(\mathcal{G})\big) & \xrightarrow{(\alpha_{X,\mathcal{G}})_0} & \mathbf{det}_X(\mathcal{G})\,. \\ f_\bullet & \longmapsto & (f_1,f_2) \end{array} \tag{58}$$

It's straightforward to deduce from the Lemma 3.6 that the function (58) is an injection. To prove that (58) is surjective we use the same Lemma and we note that for every determinant (D,T) of X with values in $\mathcal{G}$ we have that $T\big(s_i(A)\big) = s_i\big(D(A)\big)$ for every $A \in X_1$ and $0 \leq i \leq 1$. In fact take $\eta = s_i \circ s_i(A) \in X_3$ in the commutative diagram (56). ∎

Let (D_1,T_1) and (D_0,T_0) be two determinants of X with values in $\mathcal{G}$. An *homotopy from* (D_1,T_1) *to* (D_0,T_0) is by definition a triplet of functions:

$$X_1 \xrightarrow{H} \big\{\text{Objets of } \mathcal{G}\big\} \qquad \text{and} \qquad X_2 \xrightarrow{R_0,R_1} \big\{\text{Morphisms of } \mathcal{G}\big\}$$

with the following properties:

(a) For every $\xi \in X_2$ the morphisms $R_0(\xi)$ and $R_1(\xi)$ of $\mathcal{G}$ are of the type:

$$D_1(d_2\xi)\otimes H(d_0\xi) \xrightarrow{R_0(\xi)} H(d_1\xi) \xleftarrow{R_1(\xi)} H(d_2\xi)\otimes D_0(d_0\xi)\,.$$

(b) $H(s_0\star) = \mathbb{1}$.

(c) For every $\eta \in X_3$ we have the commutative diagrams of $\mathcal{G}$:

$$\begin{tikzcd}
\big(H(A_{01})\otimes D_0(A_{12})\big)\otimes D_0(A_{23}) \arrow[r, "R_1(d_3\eta)\otimes D_0(A_{23})"] \arrow[d, phantom, "a\wr\|"] & H(A_{02})\otimes D_0(A_{23}) \arrow[dd, "R_1(d_1\eta)"] & \big(D_1(A_{01})\otimes H(A_{12})\big)\otimes D_0(A_{23}) \arrow[l, "R_0(d_3\eta)\otimes D_0(A_{23})"'] \arrow[d, phantom, "\wr\| a"] \\
H(A_{01})\otimes\big(D_0(A_{12})\otimes D_0(A_{23})\big) \arrow[d, "H(A_{01})\otimes T_0(d_0\eta)"'] \arrow[r, phantom, "(\mathbf{I})"] & {} \arrow[r, phantom, "(\mathbf{II})"] & D_1(A_{01})\otimes\big(H(A_{12})\otimes D_0(A_{23})\big) \arrow[d, "D_1(A_{01})\otimes R_1(d_0\eta)"] \\
H(A_{01})\otimes D_0(A_{13}) \arrow[r, "R_1(d_2\eta)"'] & H(A_{03}) & D_1(A_{01})\otimes H(A_{13}) \arrow[l, "R_0(d_2\eta)" description] \\
{} & {} \arrow[r, phantom, "(\mathbf{III})"] & D_1(A_{01})\otimes\big(D_1(A_{12})\otimes H(A_{23})\big) \arrow[u, "D_1(A_{01})\otimes R_0(d_0\eta)"'] \arrow[d, phantom, "\wr\| a"] \\
{} & D_1(A_{02})\otimes H(A_{23}) \arrow[uu, "R_0(d_1\eta)"] & \big(D_1(A_{02})\otimes D_1(A_{12})\big)\otimes H(A_{23}) \arrow[l, "T_1(d_3\eta)\otimes H(A_{23})"]
\end{tikzcd}$$

where

$$\begin{array}{lll} A_{03} = d_1d_1\eta = d_1d_2\eta & A_{01} = d_2d_2\eta = d_2d_3\eta & A_{13} = d_1d_0\eta = d_0d_2\eta \\ A_{02} = d_2d_1\eta = d_1d_3\eta & A_{23} = d_0d_0\eta = d_0d_1\eta & A_{12} = d_2d_0\eta = d_0d_3\eta\,. \end{array}$$

Notice that given a 1-simplex of the simplicial set $\underline{\operatorname{Hom}}_{\mathbf{sSet}_0}\big(X,\mathcal{B}(\mathcal{G})\big)$, namely a map of simplicial sets $F\colon\ X\times\Delta^1 \longrightarrow \mathcal{B}(\mathcal{G})$ whose restriction to $\star\times\Delta^1$ is the constant map, then the induced pairs of functions:

$$(D_i,T_i) \;=\; \big(F_1(\,\cdot\,,\delta_i\sigma_0), F_2(\,\cdot\,,\delta_i\sigma_0\sigma_0)\big) \qquad \text{for } 0 \leq i \leq 1 \tag{59}$$

are determinants of X with values in $\mathcal{G}$, and the triplet:

$$\big(F_1(\,\cdot\,,\mathrm{id}_{[1]}),F_2(\,\cdot\,,\sigma_0),F_2(\,\cdot\,,\sigma_1)\big)$$

is an homotopy from (D_1,T_1) to (D_0,T_0).

Denote this assignment by:

$$\underline{\mathrm{Hom}}_{\mathbf{sSet}_0}\big(X,\mathcal{B}(\mathcal{G})\big)_1 \xrightarrow{(\alpha_{X,\mathcal{G}})_1} \left\{\begin{array}{c}\text{Homotopies}\\ \text{of determinants}\\ \text{from X to } \mathcal{G}\end{array}\right\} \tag{60}$$

and notice that the determinants (59) are the images by the function (58) of the following maps of simplicial sets:

$$X \cong X\times\Delta^0 \xrightarrow{X\times\delta_i} X\times\Delta^1 \xrightarrow{F} \mathcal{B}(\mathcal{G}) \qquad \text{for} \quad 0\le i\le 1.$$

4.7. Lemma. (60) *is a bijective function.*

Proof. To begin with, we note that:

$$\Delta^1_0=\Big\{\,\delta_0,\ \delta_1\,\Big\},\qquad \Delta^1_1=\Big\{\,\delta_0\sigma_0,\ \delta_1\sigma_0,\ \mathrm{id}_{[1]}\,\Big\},$$

$$\Delta^1_2=\Big\{\,\delta_0\sigma_0\sigma_0,\ \delta_1\sigma_0\sigma_0,\ \sigma_0,\sigma_1\,\Big\}\quad\text{and}\quad \Delta^1_3=\Big\{\,\delta_0\sigma_0\sigma_0\sigma_0,\ \delta_1\sigma_0\sigma_0\sigma_0,\ \sigma_0\sigma_0,\sigma_1\sigma_0,\ \sigma_1\sigma_1\,\Big\}.$$

On the other hand, for every homotopy (H,R_0,R_1) of determinants we have that $R_0(s_0A)=l_{HA}$ and $R_1(s_1A)=r_{HA}$ for every $A\in X_1$. In fact $R_1(s_1A)=r_{HA}$ follows from the diagram (**I**) of property (c) when $\eta=s_1\circ s_1(A)$ and $R_0(s_0A)=l_{HA}$ follows from (**III**) when $\eta=s_0\circ s_0(A)$.

It is easy to deduce from this and Lemma 3.6 that (60) is a bijection. ■

Since the simplicial set $\underline{\mathrm{Hom}}_{\mathbf{sSet}_0}\big(X,\mathcal{B}(\mathcal{G})\big)$ is a Kan complex it follows from Lemmas 4.6 and 4.7 that the relation induced by homotopy in the set $\mathbf{det}_X(\mathcal{G})$ is an equivalence relation. Hence if we denote by $\widetilde{\mathbf{det}}_X(\mathcal{G})$ the induced quotient set we obtain from (57) a functor:

$$\big(\mathbf{sSet}_0\big)^{op}\times 2\text{-}\mathbf{Grp} \xrightarrow{\widetilde{\mathbf{det}}_{\cdot}(\cdot)} \mathbf{Set} \tag{61}$$

such that:

4.8. Lemma. (61) *is naturally isomorphic to the functor* $\pi_0\Big(\underline{\mathrm{Hom}}_{\mathbf{sSet}_0}\big(\,\cdot\,,\mathcal{B}(\,\cdot\,)\big)\Big)$.

Let is consider the simplicial model category $\big(\mathbf{sSet}_0,\nu^\star\mathbf{W}_2,\underline{\mathrm{Hom}}_{\mathbf{sSet}_0}\big)$ of the Proposition 2.1 and recall from Corollary 3.7 that $\mathcal{B}(\mathcal{G})$ is a fibrant object of the model category $(\mathbf{sSet}_0,\nu^\star\mathbf{W}_2)$. It follows from the Lemma 4.8 that the set $\widetilde{\mathbf{det}}_X(\mathcal{G})$ is naturally isomorphic to the set of maps from X to $\mathcal{B}(\mathcal{G})$ in the homotopy category $\mathbf{sSet}_0\big[(\nu^\star\mathbf{W}_2)^{-1}\big]$ of reduced homotopy 2-types or in the homotopy category $\mathbf{sSet}_\star\big[(\pi^\star\mathbf{W}_2)^{-1}\big]$ of pointed homotopy 2-types (by the left adjoint of the adjunction (9)).

It is known that the geometric nerve functor for 2-groups induced an equivalence of categories $\widetilde{\mathcal{B}}\colon\ 2\text{-}h\mathbf{Grp} \longrightarrow \mathbf{sSet}_0\big[(\nu^\star \mathbf{W}_2)^{-1}\big]$. Let is choose a quasi-inverse functor Π_2 of this equivalence $\widetilde{\mathcal{B}}$ and for every reduced simplicial set X call the 2-group $\Pi_2(X)$ the *fundamental* 2-*group* of X.

Notice finally that for every reduced simplicial set X the functor (61) induce a functor:

$$2\text{-}h\mathbf{Grp} \xrightarrow{h\widetilde{\mathbf{det}}_X} \mathbf{Set} \tag{62}$$

from the homotopy category of 2-groups. In fact, it is a consequence of Lemma 4.8 because a monoidal natural transformation between normal lax monoidal functors $F \Rightarrow F'\colon\ \mathcal{G} \longrightarrow \mathcal{G}'$ induce an homotopy $H\colon\ \mathcal{B}(\mathcal{G}) \times \Delta^1 \longrightarrow \mathcal{B}(\mathcal{G}')$ between the morphisms $\mathcal{B}(F)$ and $\mathcal{B}(F')$ such that H restricted to $\star \times \Delta^1$ is the constant map (see for example [Noohi, 2007]).

We deduce:

4.9. Corollary. *For every reduced simplicial set X the functor* (62) *is representable by the fundamental* 2*-group of X. In particular $f\colon\ Y \longrightarrow X$ is a weak homotopy* 2*-equivalence of reduced simplicial sets if and only if the function $\widetilde{\mathbf{det}}_f(\mathcal{G})\colon\ \widetilde{\mathbf{det}}_X(\mathcal{G}) \longrightarrow \widetilde{\mathbf{det}}_Y(\mathcal{G})$ is a bijection for every* 2*-group $\mathcal{G}$.*

4.10. Let X be a reduced bisimplicial set and $\mathcal{G}$ be a 2-group. A (*functorial*) *determinant of X with values in $\mathcal{G}$* is a pair $D = (D, T)$ composed of a map of simplicial sets $D\colon\ X_{\bullet,1} \longrightarrow \mathrm{N}(\mathcal{G})$, to the geometric nerve of the underlying category of $\mathcal{G}$, and a function:

$$X_{0,2} \xrightarrow{T} \big\{\text{Morphisms of } \mathcal{G}\big\}\ ,$$

verifying the properties:

1. (Compatibility) For every $\xi \in X_{0,2}$ the morphism $T(\xi)$ of $\mathcal{G}$ is of the type:

$$D_0(d_2^v\xi) \otimes D_0(d_0^v\xi) \xrightarrow{T(\xi)} D_0(d_1^v\xi)\,.$$

2. (Functoriality) For every $\alpha \in X_{1,2}$ we have the commutative square of $\mathcal{G}$:

$$\begin{tikzcd}
D_0(d_2^v d_1^h\,\alpha) \otimes D_0(d_0^v d_1^h\,\alpha) \arrow[r, "T(d_1^h\,\alpha)"] \arrow[d, "D_1(d_2^v\alpha)\otimes D_1(d_0^v\alpha)"'] & D_0(d_1^v d_1^h\,\alpha) \arrow[d, "D_1(d_1^v\alpha)"] \\
D_0(d_2^v d_0^h\,\alpha) \otimes D_0(d_0^v d_0^h\,\alpha) \arrow[r, "T(d_0^h\,\alpha)"'] & D_0(d_1^v d_0^h\,\alpha)\,.
\end{tikzcd}$$

3. (Unit) $D_0(s_0^v\star) = \mathbb{1}$ and $T(s_0^v \circ s_0^v\star) = l_{\mathbb{1}}^{-1} = r_{\mathbb{1}}^{-1}$.

4. (Associativity) For every $\eta \in X_{0,3}$ we have the commutative diagram:

$$\begin{tikzcd}
D_0(A_{03}) & & D_0(A_{01}) \otimes D_0(A_{13}) \arrow[ll, "T(d_2^v\,\eta)"'] \\
 & & D_0(A_{01}) \otimes \big(D_0(A_{12}) \otimes D_0(A_{23})\big) \arrow[u, "D_0(A_{01})\otimes T(d_0^v\,\eta)"'] \\
D_0(A_{02}) \otimes D_0(A_{23}) \arrow[uu, "T(d_1^v\,\eta)"] & & \big(D_0(A_{01}) \otimes D_0(A_{12})\big) \otimes D_0(A_{23})\,, \arrow[ll, "T(d_3^v\,\eta)\otimes D_0(A_{23})"] \arrow[u, phantom, "a\wr\|"]
\end{tikzcd}$$

where

$$\begin{array}{llllll}
A_{03} = d_1^v d_1^v \eta = d_1^v d_2^v \eta & A_{01} = d_2^v d_2^v \eta = d_2^v d_3^v \eta & A_{13} = d_1^v d_0^v \eta = d_0^v d_2^v \eta \\
A_{02} = d_2^v d_1^v \eta = d_1^v d_3^v \eta & A_{23} = d_0^v d_0^v \eta = d_0^v d_1^v \eta & A_{12} = d_2^v d_0^v \eta = d_0^v d_3^v \eta\,.
\end{array}$$

Let is denote by $\underline{\mathbf{det}}_X(\mathcal{G})_0$ the set of the determinants of X with values in $\mathcal{G}$. There is a functor:

$$\big(\mathbf{ssSet}_0\big)^{op} \times 2\text{-}\mathbf{Grp} \xrightarrow{\underline{\mathbf{det}}_{\,\cdot}(\,\cdot\,)_0} \mathbf{Set} \tag{63}$$

defined in the morphisms $f\colon\ Y \longrightarrow X$ and $F = (F, m^F)\colon\ \mathcal{G} \longrightarrow \mathcal{H}$ as $\underline{\mathbf{det}}_f(F)(D,T) = (\overline{D}, \overline{T})$ where $\overline{D} = \mathrm{N}(F) \circ D \circ f_{\bullet,1}$ and $\overline{T} = \mathcal{N}(F)_2 \circ T \circ f_{0,2}$.

Denote by $\mathcal{B}^2\colon\ 2\text{-}\mathbf{Grp} \longrightarrow \mathbf{ssSet}_0$ the geometrical 2-nerve functor for monoidal categories (33) when it is restricted to the category of 2-groups.

4.11. Lemma. *(63) is naturally isomorphic to the functor* $\mathrm{Hom}_{\mathbf{ssSet}_0}\big(\,\cdot\,,\, \mathcal{B}^2(\,\cdot\,)\big)$.

Proof. It follows from 3.10 that the following function is a bijection:

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathbf{ssSet}_0}\big(X, \mathcal{B}^2(\mathcal{G})\big) & \xrightarrow{(\alpha^2_{X,\mathcal{G}})_0} & \underline{\mathbf{det}}_X(\mathcal{G})_0 \\
f_{\bullet,\bullet} & \longmapsto & (f_{\bullet,1}, f_{0,2})
\end{array} \tag{64}$$

■

Given $\big(D(1), T(1)\big)$ and $\big(D(0), T(0)\big)$ two determinants of X with values in $\mathcal{G}$, an *homotopy from* $\big(D(1), T(1)\big)$ *to* $\big(D(0), T(0)\big)$ is a function:

$$X_{0,1} \xrightarrow{h} \big\{\text{Morphisms of } \mathcal{G}\big\}$$

with the following properties:

1. For every $a \in X_{0,1}$ the morphism $h(a)$ of $\mathcal{G}$ is of the type: $D(1)_0(a) \xrightarrow{h(a)} D(0)_0(a)$.
2. For every $\alpha \in X_{1,1}$ we have a commutative diagram of $\mathcal{G}$:

$$\begin{tikzcd}
D(1)_0(d_1^h\alpha) \arrow[r, "D(1)_1(\alpha)"] \arrow[d, "h(d_1^h\alpha)"'] & D(1)_0(d_0^h\alpha) \arrow[d, "h(d_0^h\alpha)"] \\
D(0)_0(d_1^h\alpha) \arrow[r, "D(0)_1(\alpha)"'] & D(0)_0(d_0^h\alpha)
\end{tikzcd}$$

3. $h(s_0^v\star) \;=\; \mathrm{id}_{\mathbb{1}}$.

4. For every $\xi \in X_{0,2}$ we have a commutative diagram of $\mathcal{G}$:

$$\begin{tikzcd}
D(1)_0(d_2^v\xi)\otimes D(1)_0(d_0^v\xi) \arrow[r, "T(1)(\xi)"] \arrow[d, "h(d_2^v\xi)\otimes h(d_0^v\xi)"'] & D(1)_0(d_1^v\xi) \arrow[d, "h(d_1^v\xi)"] \\
D(0)_0(d_2^v\xi)\otimes D(0)_0(d_0^v\xi) \arrow[r, "T(0)(\xi)"'] & D(0)_0(d_1^v\xi)
\end{tikzcd}$$

It is easy to see that there is a well defined function:

$$\begin{array}{ccc} \underline{\mathrm{Hom}}_{\mathbf{ssSet}_0}\big(X,\mathcal{B}^2(\mathcal{G})\big)_1 & \xrightarrow{(\alpha^2_{X,\mathcal{G}})_1} & \left\{\begin{array}{c}\text{Homotopies}\\ \text{of determinants}\\ \text{from X to }\mathcal{G}\end{array}\right\} \\ H & \longmapsto & H_{1,1}\left(s_0^h(\,-\,),\,\mathrm{id}_{[1]}\right) \end{array} \tag{65}$$

where $H_{1,1}\left(s_0^h(\,-\,),\,\mathrm{id}_{[1]}\right)$ is an homotopy from $\big(H_{\bullet,1}(\,-\,,\,\delta_1\underbrace{\sigma_0\cdots\sigma_0}_{\bullet}),\,H_{0,2}(\,-\,,\,\delta_1)\big)$ to $\big(H_{\bullet,1}(\,-\,,\,\delta_0\underbrace{\sigma_0\cdots\sigma_0}_{\bullet}),\,H_{0,2}(\,-\,,\,\delta_0)\big)$.

It is not difficult to show:

4.12. LEMMA. (65) *is a bijective function.*

Let is notice:

4.13. LEMMA. *If $\mathcal{G}$ is a 2-group and X is a reduced bisimplicial set, the simplicial set $\underline{\mathrm{Hom}}^{(1)}_{\mathbf{ssSet}_0}\Big(X,\mathcal{B}^2(\mathcal{G})\Big)$ is the nerve of a groupoid. In particular, there exist a functor:*

$$\big(\mathbf{ssSet}_0\big)^{op}\times 2\text{-}\mathbf{Grp} \xrightarrow{\underline{\mathbf{det}}_{\cdot}(\,\cdot\,)} \mathbf{Grpd}\,, \tag{66}$$

where **Grpd** *is the category of groupoids and functors, with the properties:*

1. *The composition of (66) with the "set of objects" functor* $\mathbf{Grpd}\to\mathbf{Set}$ *is equal to (63).*
2. *The set of morphism of the groupoid $\underline{\mathbf{det}}_X(\mathcal{G})$ is equal to the set of homotopies of determinants.*
3. *The simplicial sets $\mathrm{N}\big(\underline{\mathbf{det}}_X(\mathcal{G})\big)$ and $\underline{\mathrm{Hom}}^{(1)}_{\mathbf{ssSet}_0}\Big(X,\mathcal{N}^2(\mathcal{G})\Big)$ are naturally isomorphic.*

Proof. It's not difficult to prove that the simplicial set $\underline{\mathrm{Hom}}_{\mathbf{sSet}}\big(A, \mathrm{N}(\mathcal{G})\big)$ is the nerve of a groupoid for every simplicial set A and every groupoid $\mathcal{G}$. On the other hand, notice that the simplicial set $\underline{\mathrm{Hom}}^{(1)}_{\mathbf{ssSet}_0}\big(X, \mathcal{B}^2(\mathcal{G})\big)$ is the kernel in the category $\mathbf{sSet}$ of the arrows:

$$\begin{tikzcd} \prod\limits_{q\geq 0} \underline{\mathrm{Hom}}_{\mathbf{sSet}}\big(X_{\bullet,q}, \mathcal{B}^2(\mathcal{G})_{\bullet,q}\big) \arrow[r, shift left, "\prod_{\varphi} \mathcal{B}^2(\mathcal{G})_{\varphi} \circ \mathrm{proj}_t"] \arrow[r, shift right, "\prod_{\varphi} \mathrm{proj}_s \circ X_{\varphi}"'] & \prod\limits_{\varphi\colon s\to t} \underline{\mathrm{Hom}}_{\mathbf{sSet}}\big(X_{\bullet,t}, \mathcal{B}^2(\mathcal{G})_{\bullet,s}\big) \end{tikzcd} .$$

where $\mathcal{B}^2(\mathcal{G})_{\bullet,q}$ is the nerve of a groupoid. ∎

Let is consider the simplicial model category $\big(\mathbf{ssSet}_0, d^\star\mathbf{W}_2, \underline{\mathrm{Hom}}^{(1)}\big)$ of the Proposition 2.9. By the Theorem 3.9 the reduced bisimplicial set $\mathcal{B}^2(\mathcal{G})$ is a fibrant object of the model category $(\mathbf{ssSet}_0, d^\star\mathbf{W}_2)$, then from the Lemma 4.13 the nerve of the groupoid $\underline{\mathbf{det}}_X(\mathcal{G})$ is of the same homotopy type as the mapping space from X to $\mathcal{B}^2(\mathcal{G})$ in the model category $(\mathbf{ssSet}_0, d^\star\mathbf{W}_2)$. It follows (see the adjunctions (21)) that the nerve of the groupoid $\underline{\mathbf{det}}_X(\mathcal{G})$ is also of the same homotopy type as the mapping space from $diag(X)$ to $\mathcal{B}(\mathcal{G})$ in the model category $(\mathbf{sSet}_0, v^\star\mathbf{W}_2)$.

Denote $h\mathbf{Grpd}$ the *homotopy category of groupoids*, that is to say the category that we obtain from $\mathbf{Grpd}$ by identifying two functors if they are naturally isomorphic. It follows from the Corollary 3.14 and Lemma 4.13 that the functor (66) induce a functor:

$$2\text{-}h\mathbf{Grp} \xrightarrow{h\underline{\mathbf{det}}_X(\,\cdot\,)} h\mathbf{Grpd} \tag{67}$$

for every reduced bisimplicial set X.

4.14. Corollary. *For every reduced bisimplicial set X the functor (67) is representable by the fundamental 2-group of the reduced simplicial set $diag(X)$. In particular a map of reduced bisimplicial sets $f\colon\ Y \longrightarrow X$ is a diagonal weak homotopy 2-equivalence if and only if the functor $\underline{\mathbf{det}}_f(\mathcal{G})\colon\ \underline{\mathbf{det}}_X(\mathcal{G}) \longrightarrow \underline{\mathbf{det}}_Y(\mathcal{G})$ is a weak homotopy equivalence of groupoids for every 2-group $\mathcal{G}$.*

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